HONORS ALGORITHMS

2024-02-16

PROB "Finite Probability Spaces"

"Last updated Feb 15, 2024"

last night

Finite probability space $(\Omega, \Pr)$

Ω finite set, $\Omega \neq \emptyset$

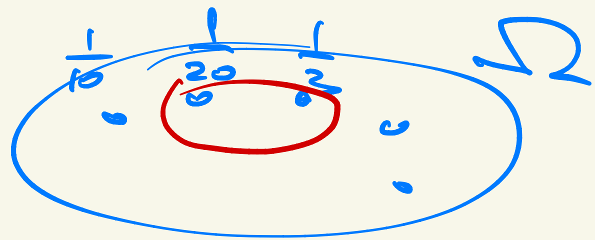
$\Pr : \Omega \to \mathbb{R}$ probability distribution on Ω

(i) $(\forall a \in \Omega)(\Pr(a) \geq 0)$

(ii) $\sum_{a \in \Omega} \Pr(a) = 1$

Event: $A \subseteq \Omega$

$\Pr(A) := \sum_{a \in A} \Pr(a)$



RANDOM VARIABLE

$$X : \Omega \to \mathbb{R}$$

set of random variables

$$\mathbb{R}^{\Omega}$$

we can take lin. comb.

$\rightsquigarrow \mathbb{R}^{\Omega}$ is a <u>vector space</u> $\quad \dim = |\Omega|$

a basis: $\quad 0\ 0\ 1\ 0\ 0\ 0\ 0$

$\quad \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$

Notation:

$$\{f : A \to B\} = B^{A}$$

$$|B^{A}| = |B|^{|A|}$$

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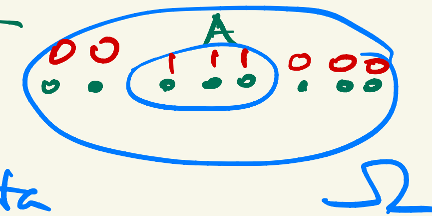
indicator variable: r.v. codomain $\{0,1\}$

$$Y : \Omega \longrightarrow \{0,1\}$$



$\longleftrightarrow$ with events

1-1 corr.

\vartheta

$$\vartheta_A : \Omega \to \mathbb{R} \qquad \vartheta_A(a) = \begin{cases} 1 & a \in A \\ 0 & o/w \end{cases}$$

$A \subseteq \Omega$

indicator of event A

inverse corresp: if Y is an indicator var.

then $Y = \vartheta_A$ where $A = Y^{-1}(1)$

$$\Omega \left\{ \begin{array}{c|c} \Omega & X \\ \hline 1 & X(1) \\ \vdots & \vdots \\ n & X(n) \end{array} \right.$$

$|\Omega|$ data

Ω	Pr	X
a	$1/2$	5
b	$1/3$	1
c	$1/6$	-10

$$5 \cdot \frac{1}{2} + 1 \cdot \frac{1}{3} - 10 \cdot \frac{1}{6}$$

1st aggregate of data:

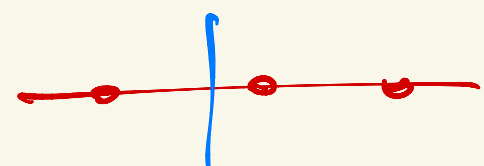
EXPECTED VALUE

$$\rightarrow \quad E(X) = \sum_{a \in \Omega} X(a) \cdot Pr(a)$$

weighted average of the values of X

If Pr is uniform:
$n = |\Omega|$

$$E(X) = \frac{1}{n} \sum_{a \in \Omega} X(a)$$



Biased coin $\quad P(\text{heads}) = p$

$\quad\quad\quad\quad\quad\; P(\text{tails}) = 1-p$

$$|\Omega| = 2^n$$

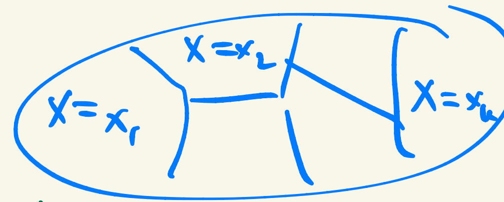
flip n times

$$|\text{Range } X| = n+1$$

$$X := \#\text{ heads}$$

$\boxed{\text{DO}}$

$$E(X) = \sum_{x_i \in \text{Range}(X)} x_i \, P(X = x_i)$$



$$\text{"}X = x_i\text{" event} = \{a \in \Omega \mid X(a) = x_i\}$$

$$= \sum_{y \in \mathbb{R}} y \, P(X = y)$$

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$$E(\vartheta_A) = 1 \cdot \Pr(\vartheta_A = 1) + 0 \cdot \Pr(\vartheta_A = 0) = \Pr(A)$$

$$\text{"}\vartheta_A = 1\text{"} = \{a \in \Omega \mid \vartheta_A(a) = 1\} = A$$

$$\boxed{E(\vartheta_A) = \Pr(A)}$$

$p/1-p$ biased coins X : # heads in n coin flips

$$E(X) = np$$

$$E(X) = \sum_{y=0}^{n} y \cdot P(X=y)$$

$$\sum_{y=0}^{n} y \cdot \binom{n}{y} p^{y} \cdot (1-p)^{n-y} = \sum_{y=1}^{n} \text{same} =$$

$$y \cdot \binom{n}{y} = y \cdot \frac{n(n-1)\cdots(n-y+1)}{y!} = n \cdot \binom{n-1}{y-1}$$

$$= np \cdot \underbrace{\sum_{y=1}^{n} \binom{n-1}{y-1} \cdot p^{y-1} \cdot (1-p)^{(n-1)-(y-1)}}_{(p+(1-p))^{n-1} = 1^{n-1} = 1} = np$$

$X = Y_1 + \cdots + Y_n$

Y_i: indicates event

i^{th} coin : heads

$$\boxed{E(X) = \sum_{i=1}^{n} \underbrace{E(Y_i)}_{=\Pr(i^{th}\text{ coin heads}) = p} = np} \checkmark$$

Do $E(\sum X_i) = \sum E(X_i)$

LINEARITY OF EXPECTATION

X_i : any r.v.'s

Do
$$\boxed{E(\sum c_i X_i) = \sum c_i E(X_i)}$$

Boolean variable $x \in \{0, 1\}$ → 's

$x_1 \cdots x_n$

$\bar{x}_i = 1 - x_i$

$x_1 \cdots x_n \quad \bar{x}_1 \cdots \bar{x}_n$: literals

CONJUNCTION : $a_1 \wedge \cdots \wedge a_m$

DISJUNCTION : $a_1 \vee \cdots \vee a_m$

3-DISJ. $a_1 \vee a_2 \vee a_3$ where the a_i are <u>literals</u>

$C_1, \ldots, C_m$ 3-disjunctions

find assignment (0,1) to variables x_i s.t.

$$\max \leftarrow \#\{i \mid C_i(x_1 \cdots x_n) = 1\}$$

At least $\frac{7}{8}m$ can be simultaneously satisfied

Claim X : # satisfied clauses under uniform random substitution $\left(\frac{1}{2}, \frac{1}{2}\right)$

then $E(X) = \frac{7m}{8}$