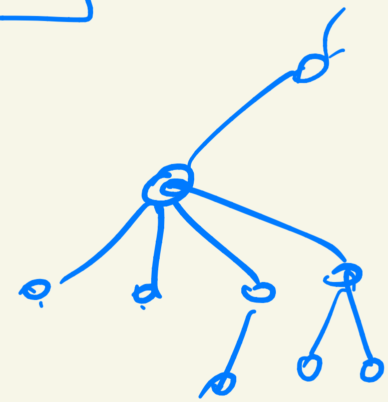
PROBLEM SESSION

2024-02-16

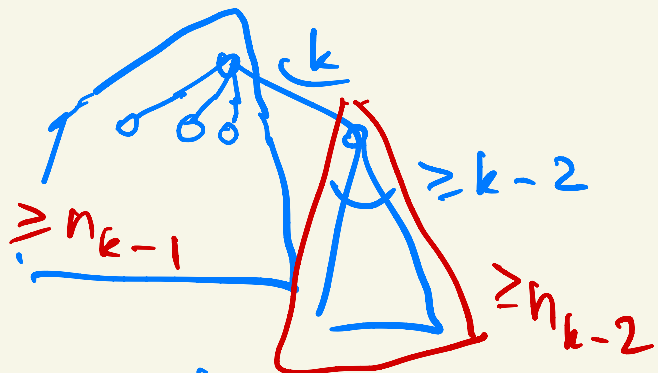
13.34 FREDMAN-TARJAN tree:

\# children of i^{th} child $\geq i-2$



given root degree

min # vertices n_k



$n_k \geq n_{k-1} + n_{k-2}$

$n_k = n_{k-1} + n_{k-2}$

$k=0$ o $\quad$ $k=1$

$n_0 = 1$ $\quad$ $n_1 = 2$

$$\boxed{n_k = F_{k+2}}$$

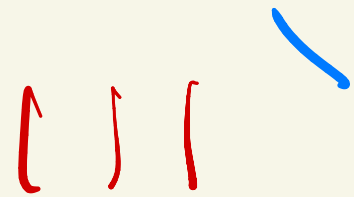
by induction

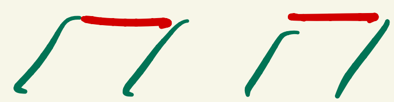
13.44 Maximal vs. maximum matching

M maximal $\qquad \nu = \nu(G)$ size of maximum matching

Claim $\quad |M| \geq \frac{\nu}{2}$

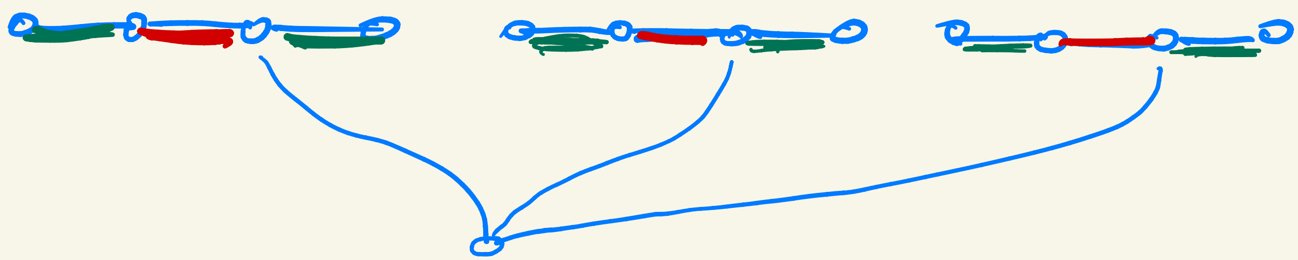
b/c M must hit every edge



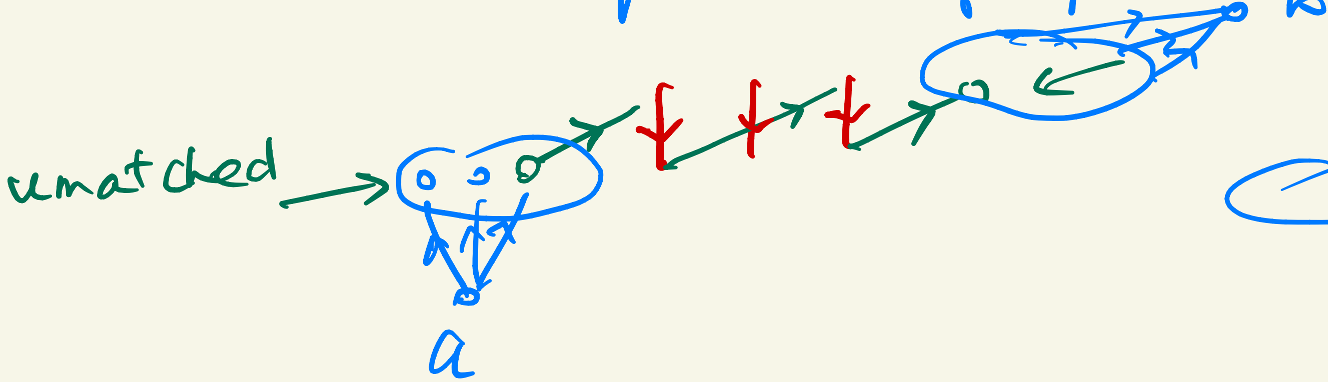


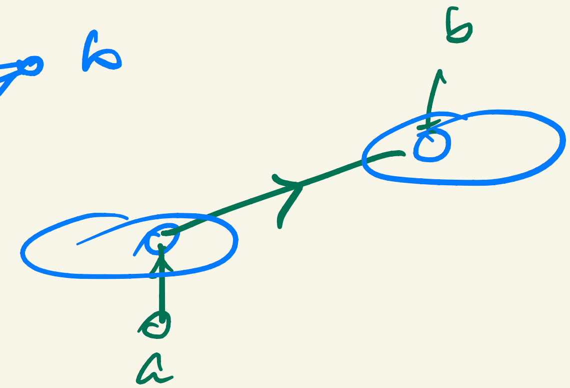
tight for conn. bipartite graphs
for every $k \qquad \nu = 2k$

P_3



13.56 find augmenting path in bipartite graph





w

v

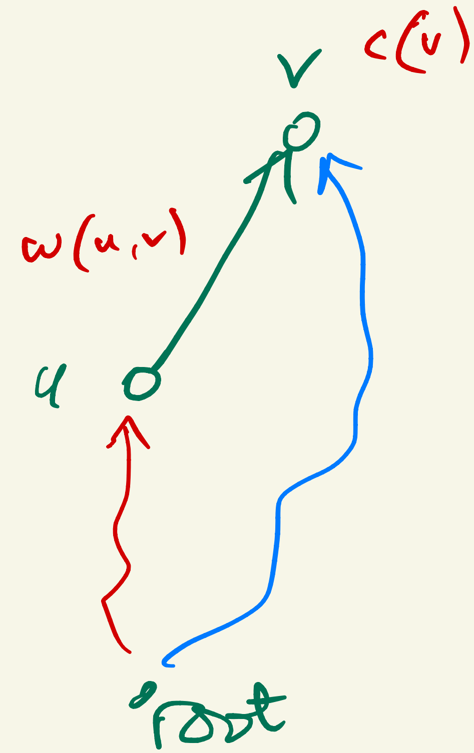
color = dist(v, w) mod 2

13.94

	JARNÍK	DIJKSTRA
goal	spanning tree	min-weight paths
input	graph no root	digraph root
	edge-weighted	edge-weighted
output	parent links after selecting a root	parent links

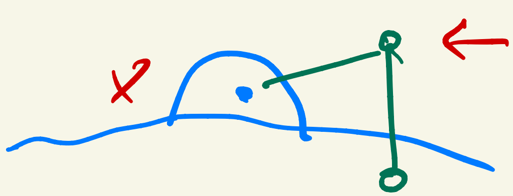
RELAX $c(v) \leftarrow \min\big(c(v),\ c(u)+w(u,v)\big)$

DIJKSTRA $c(v)$ min weight of root $\to v$ path through black vertices



JARNÍK $c(v) \leftarrow \min\big(c(v),\ w(u,v)\big)$

$c(v)$ min cost of black ~~x~~-neighbor



14.51 G graph $\underline{\chi(G)\cdot\alpha(G) \geq n}$

$k = \chi(G)$ $V = C_1 \sqcup \cdots \sqcup C_k$

C_i is indep.

$\therefore$ $|C_i| \leq \alpha$

$n \leq \alpha \cdot k$ ✓

14.74 α (regular graph)

degree $= d \geq 1$

Claim $\alpha \leq \frac{n}{2}$

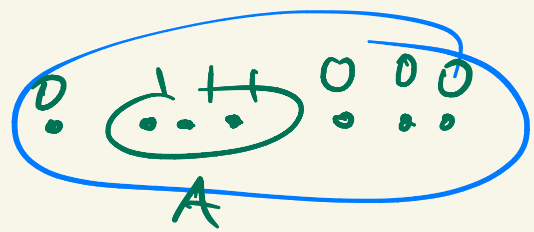
Pf: $A \subseteq V$ A indep. NTS $|A| \leq \frac{n}{2}$

charact. fcn of A

vertex $i \mapsto x_i$

$x_i \geq 0$

if $i \sim j$ then $x_i + x_j \leq 1$ $\leftarrow$ m inequalities



adding them up: $\sum_{i \sim j} (x_i + x_j) \leq m = \frac{nd}{2}$

$d|A| = d \cdot (\sum x_i)$

$d \cdot |A| \leq \frac{nd}{2}$ $\div d$ b/c $d \neq 0$

14.6 | $\underbrace{G \not\supset K_3}_{\Downarrow} \implies \chi(G) = O(\sqrt{n})$

$(\forall v)(N(v)\ \text{indep})$

while ∃ vertex of deg $\geq \sqrt{n}$
    color its neighbors with a new color
    delete $N(v)$
end (while)
    finish using $< \sqrt{n}+1$ colors

repeat $\leq \sqrt{n}$ times

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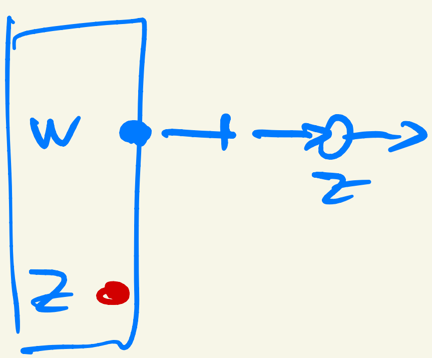
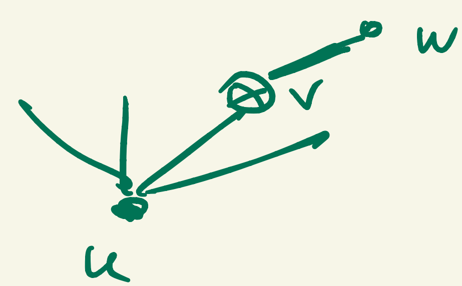
for $v \in V$
  for $w \in \operatorname{adj} V$
    $\deg(v) \leftarrow \deg(v)+1$



$2\sqrt{n}+1$

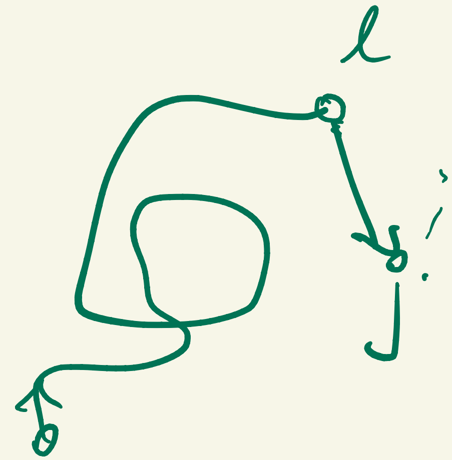


14.79 counting triangles $= \overset{6}{c} \cdot \operatorname{tr}(A^3)$

$A^k = (a_{ij}^{(k)})$ $\qquad a_{ij}^{(k)} = \#\ i \to \cdots \to j$ walks

Pf: induction on k

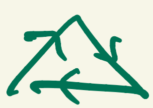
$$a_{ij}^{(k)} = \sum_{l=1}^{n} a_{il}^{(k-1)} \cdot a_{lj}$$



$a_{ij}^{(3)}$: # walks of length 3

$i \to \ \to \ \to j$

$a_{ii}^{(3)}$:



$\sum a_{ii}^{(3)}$ counts every Δ 6 times

Computing A^3 in bit model

STRASSEN

A^2 n^β arithm. ops $\beta = \log_2 7$

$\downarrow$

on $O(\log n)$-bit numbers

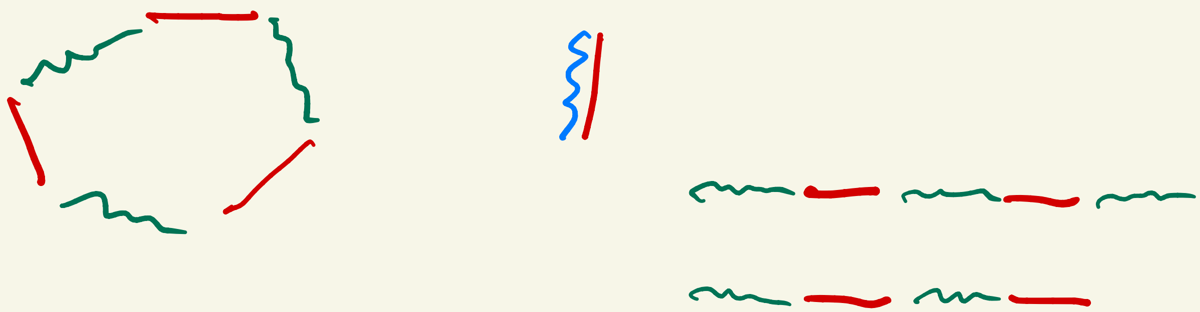
total $O(n^\beta \cdot \log n)$

multipl: $a \cdot 1$

i.e. only additions

13.75 approximation ratio of greedy weighted matching

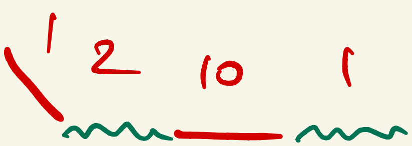
Claim greedy $\geq \frac{1}{2}$ OPT

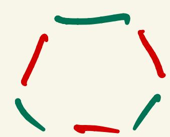


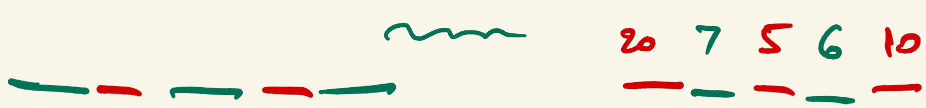
$M \cup OPT$

Lemma Every ~~red~~ green (OPT) edge has a ~~green~~ red (greedy) neighbor

$w(\text{red}) \geq w(\text{green neighbor})$



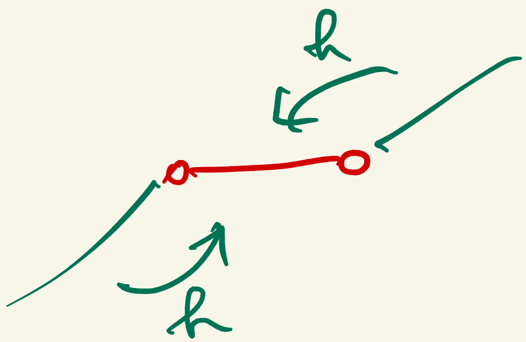




LEMMA $e \notin M \Rightarrow \exists\, \underset{f \in M}{f \neq e}$ s.t. $w(f) \geq w(e)$

$e \in OPT \quad h(e) \in M$ s.t. $w(h(e)) \geq w(e)$

$$2 \sum_{f \in M} w(f) \geq \sum_{e \in OPT} w(e)$$

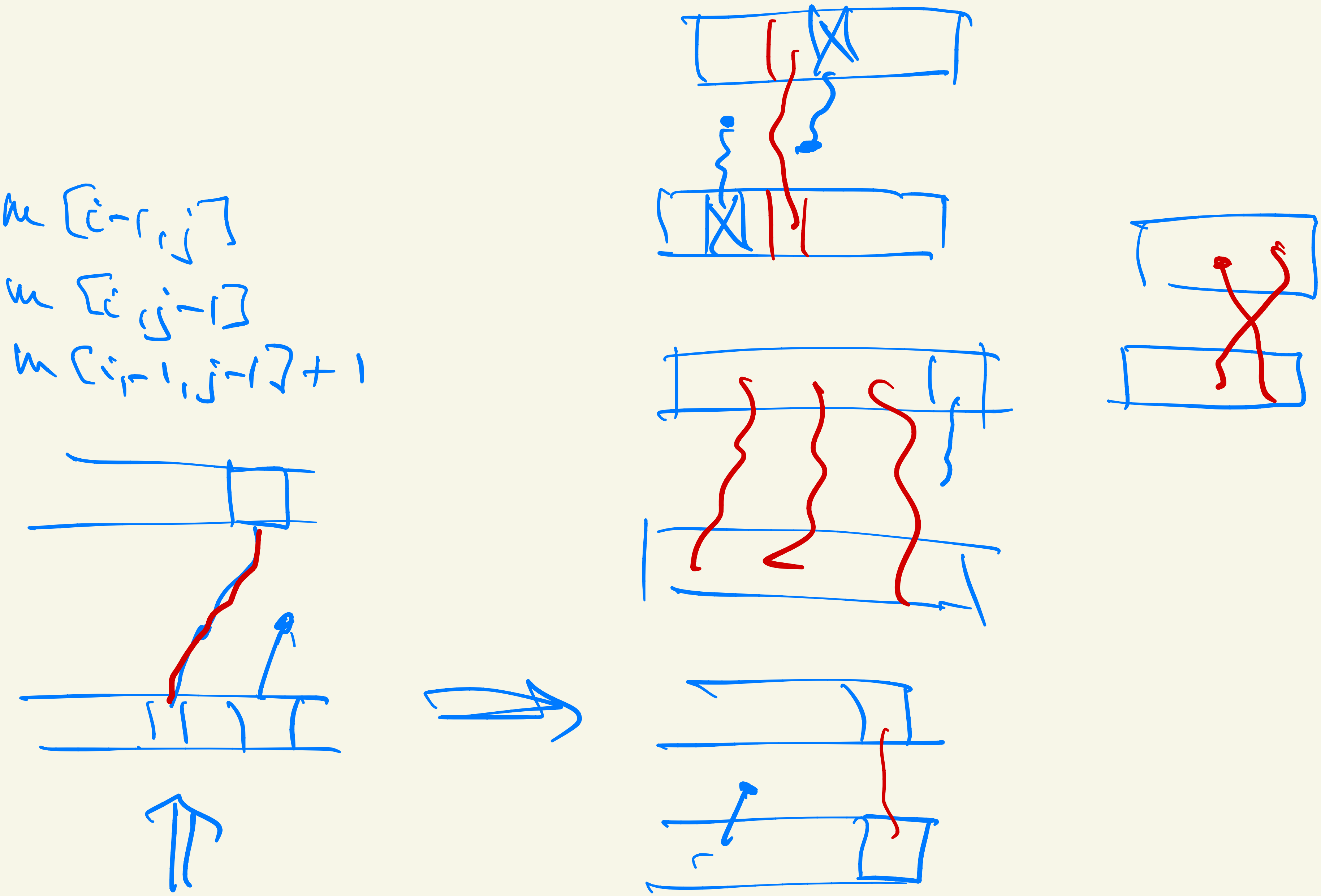


Edit distance

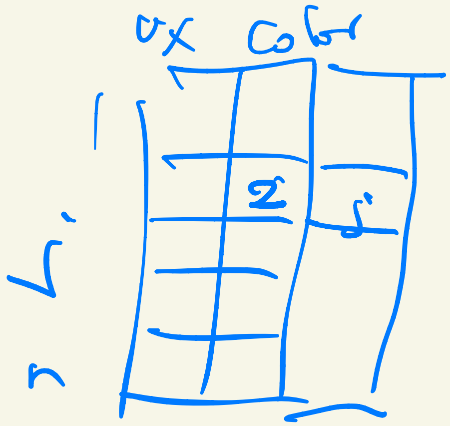
$m[i-1, j]$

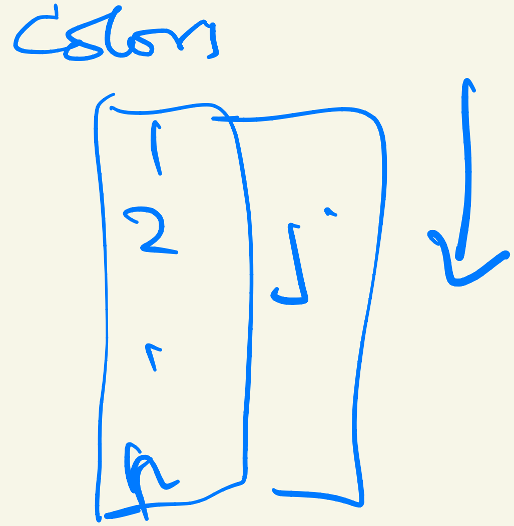
$m[i, j-1]$

$m[i-1, j-1] + 1$



greedy coloring in linear time





subfield

$F \subset G$

$A \in F^{k \times l}$

$\Longrightarrow \mathrm{rk}_F(A) = \mathrm{rk}_G(A)$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$