HONORS ALGORITHMS | 2024-02-19 | 1

Conditional probability

$$P(A \mid B) := \frac{P(A \cap B)}{P(B)}$$

B ↑ given

Theorem of complete probability

$\Omega = H_1 \cup \dots \cup H_k$ partition

$(\forall i \neq j)(H_i \cap H_j = \emptyset)$, $P(H_j) \neq 0$

$$\boxed{\forall A \subseteq \Omega \quad P(A) = \sum_i P(A \mid H_i) \cdot P(H_i)}$$

$$P(A) = \sum_i P(\underbrace{A \cap H_i}_{\text{these are disjoint}}) =$$

Theorem of complete probability for events

$\Omega = H_1 \cup \dots \cup H_k$ partition

$(\forall i \neq j)\,(H_i \cap H_j = \emptyset)$, $P(H_j) \neq 0$

$$\boxed{\begin{array}{l}\forall A \subseteq \Omega \\ P(A) = \sum_i P(A \mid H_i) \cdot P(H_i)\end{array}}$$

$$P(A) = \sum_i P(\underbrace{A \cap H_i}_{\text{these are disjoint}}) =$$

$$E(X) = \sum_{a \in \Omega} X(a) \cdot P(a)$$

$X : \Omega \to \mathbb{R}$

$B \subseteq \Omega$

$P(B) \neq 0$

$$E(X \mid B) = \frac{\sum_{a \in B} X(a) \cdot P(a)}{P(B)}$$

Thm of complete prob for r.v's

$$E(X) = \sum_i E(X \mid H_i) \cdot P(H_i)$$

$P = Pr$

Pf $E(X) = \sum_i \underbrace{\sum_{a \in H_i} X(a) \cdot P(a)} = \sum_i E(X \mid H_i)\, P(H_i)$

Independence of r.v.'s

(fully)

$X, Y : \Omega \to \mathbb{R}$ are independent if

$$(\forall x, y \in \mathbb{R})\left(P(X = x \wedge Y = y) = P(X = x) \cdot P(Y = y)\right)$$

DEF $X_1, \dots, X_k : \Omega \to \mathbb{R}$ are independent if

$$(\forall x_1 \dots x_k \in \mathbb{R})\left(P\left(\bigwedge_{i=1}^{k} X_i = x_i\right) = \prod_{i=1}^{k} P(X_i = x_i)\right)$$

$\boxed{\text{DO}}$ If $X_1 \dots X_k$ are independent then

$$(\forall I \subseteq [k])(X_i \mid i \in I) \text{ are indep.}$$

$\boxed{\text{DO}}$ Events $A_1 \dots A_k$ are indep $\iff \vartheta_{A_1} \dots \vartheta_{A_k}$ indep.

indicator variables

$\boxed{\text{DO}}$ **Thm** If X, Y are indep.

then $E(XY) = E(X)E(Y)$

$X, Y : \Omega \to \mathbb{R}$

Same prob. space

$\boxed{\text{EX}}$ Converse false

DEF X, Y are uncorrelated if $E(XY) = E(X)E(Y)$

positively correlated if $>$

negatively " if $<$

find X, Y that uncorrelated but not independent

Thm If $X_1 \ldots X_k$ are indep. r.v's then

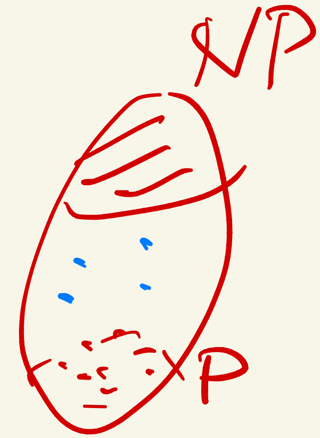
$$E\left(\prod_i X_i\right) = \prod_i E(X_i)$$

$$E\left(\sum X_i\right) = \sum E(X_i) \longleftarrow \text{unconditionally}$$

3-COL : 3-colorability

input : graph
question: is it 3-colorable

} decision problem



HAM : Hamiltonicity

input : graph
question : is it Hamiltonian, i.e.
$\exists$ Hamilton cycle : passes through all vertices

CLIQUE : input : (graph G, k integer)

question : $\omega(G) \geq k$

i.e. $\exists K_k \subseteq G$

these are **NP-problems**

"nondeterministic polynomial time"

P : class of decision problems
decidable in <u>polynomial time</u>

with respect to bit-length of input

NP : class of decision problems

$$A(\underline{x}) = Y/N$$

$$A(x) \longleftrightarrow (\exists y)(B(\underline{x},\underline{y}))$$

polynomial-time decision problem

$$|y| \leq |x|^c$$

[Do] $NP \supseteq P$

$$P \neq NP$$