HONORS ALGORITHMS | 2024-02-23 | 1

COOK REDUCTION

$f_i : \Sigma_i^* \to \Pi_i^*$

$$f_1 \prec_{\text{cook}} f_2$$

oracle ↓

p-time alg for f_1 uses queries to f_2

If f_2 can be solved in poly time
then f_1 — " — ____

Karp reduction

$L_1 \prec_{\text{Karp}} L_2$

$L_i \subseteq \Sigma_i^*$

$f : \Sigma_1^* \to \Sigma_2^*$

(i) f computable in P-time

(ii) $(\forall x \in \Sigma_1^*)(x \in L_1 \leftrightarrow f(x) \in L_2)$

E.g. $3COL \prec_{\text{Karp}} HAM$

factoring integers:

INPUT: $m \in \mathbb{N}$

OUTPUT: list of prime factors of m

DECISION VERSION – language

$\text{FACTOR} = \{(m,k) \mid m,k \in \mathbb{N}, (\exists d \in \mathbb{N})(2 \le d \le k \wedge d \mid m)\}$

FACTOR $\in$ NP witness: d

FACTOR $\prec_{\text{Cook}}$ factoring (YES $\leftrightarrow$ smallest prime factor $\le k$)

Thm factoring $\prec_{\text{Cook}}$ FACTOR

Pf. Use binary search to find smallest prime factor p

repeat for $\frac{m}{p}$

$\boxed{\text{DO}}$ Analyze this algorithm

DEF $L \subseteq \Sigma^*$ is NP-complete

if (i) $L \in NP$

(ii) $(\forall M \in NP)(M \prec_{KARP} L)$

$$\text{coNP} = \{L \mid L \subseteq \Sigma^*,\ \Sigma^* \setminus L \in NP\}$$

FACTOR $\in$ coNP

$\therefore$ If FACTOR $\in$ NPC $\leftarrow$ NP-complete

then $NP = coNP$

COOK–LEVIN THM (1972)

there exist natural examples of languages that are NP-complete

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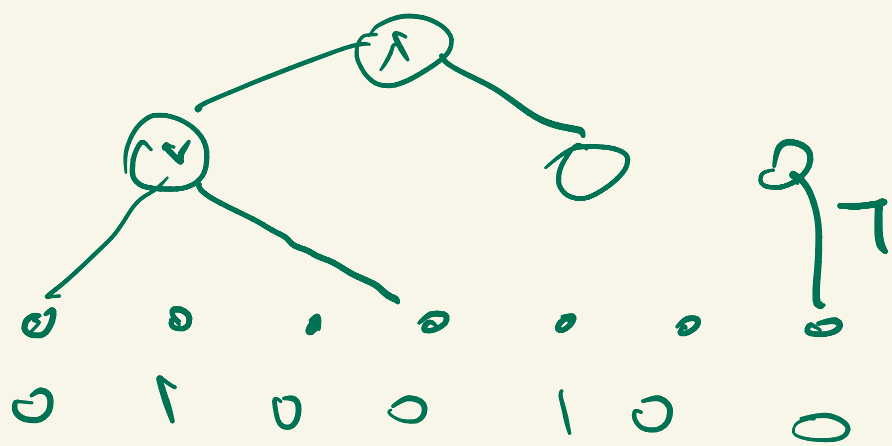
Boolean circuit satisfiability SAT

INPUT



Boolean function $f : \{0,1\}^n \rightarrow \{0,1\}$

EX $\forall$ Boolean fctn can be computed by a Boolean circuit

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CNF conjunctive normal form

$C_1, \dots, C_m$ disjunctive clauses: $\underbrace{x_3 \vee \overline{x_5} \vee x_8 \vee x_{10}}$

$$\bigwedge_{i=1}^{m} C_i$$

EX $\forall$ Boolean fctn can be repr. as CNF

SAT = {satisfiable CNFs} $\in NP$

$$(\exists \underline{a} \in \{0,1\}^n)(f(\underline{a}) = 1)$$

3CN-Satisfiability : **3SAT**

Thm 3SAT $\in NPC$

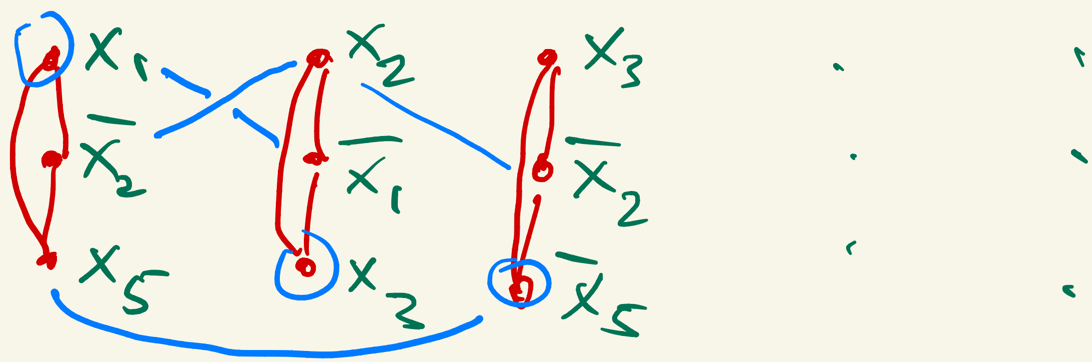
COR CLIQUE $\in$ NPC

$$\text{CLIQUE} = \{(G, k) \mid G \text{ has a } k\text{-clique}\}$$

Proof (i) CLIQUE $\in$ NP ✓

(ii) $\text{3SAT} \prec_{\text{KARP}} \text{CLIQUE}$

G



Claim

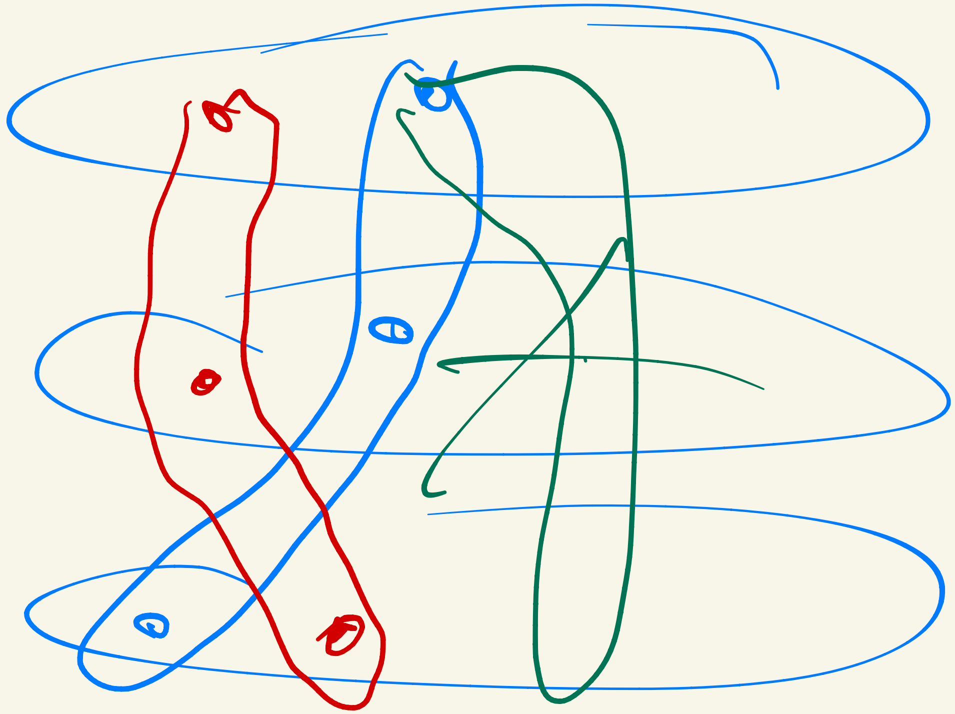
$\alpha(G) = m$

$\Updownarrow$

f is satisf.

$n = 3m$ vertices
↳ #clauses

$\alpha(G) \leq m$



3D-matching $\in$ NPC