PROBLEM SESSION

2024-03-01

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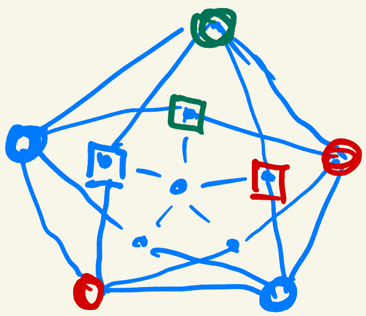
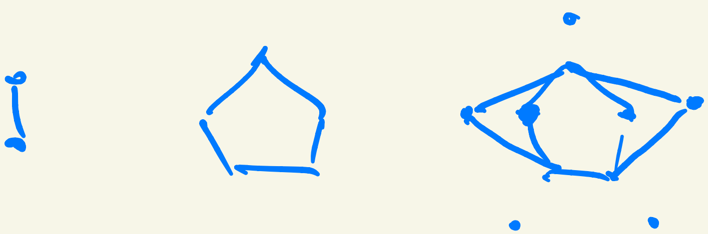
14.67 $G \not\supset K_3$, $\chi(G) \geq 4$, $n = 11$, 5-fold symmetry



GRÖTZSCH'S GRAPH

$\downarrow$

MYCZIELSKI's graphs

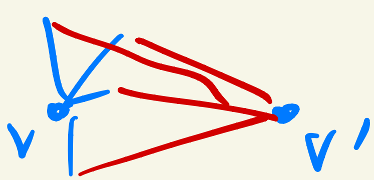


$G_k \not\supset \Delta$

$\chi(G_k) \geq k$

$|V(G_k)|$ exponential $> 2^k$

DO G + ghost of G



ERDŐS 1957: k^c vertices suffice

EXPLICIT: Ramanujan graphs

PROBABILISTIC METHOD

19.34 $E(\underbrace{\#\text{Aces}}_{Y})$ in poker hand

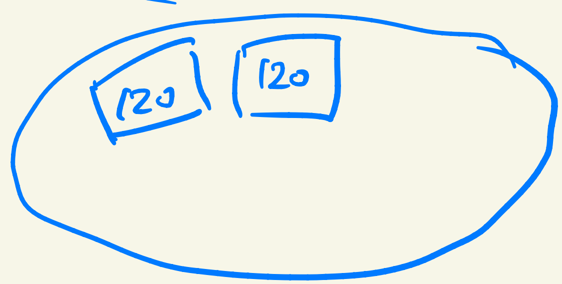
Sol#1 X_i : indicator that i^{th} card is an Ace

$$Y = \sum_{i=1}^{5} X_i$$

$$E(Y) = \sum E(X_i) = \sum \underbrace{P(i^{th}\text{ card is Ace})}_{\frac{1}{13}} = \frac{5}{13}$$

$\binom{52}{5}$ ✗ no 1^{st} card etc.

$$\binom{52}{5}\cdot 5! = 52\cdot 51\cdot 50\cdot 49\cdot 48$$



$$E_1(Y) = \frac{z_1 + z_2 + \cdots}{\binom{52}{5}}$$

first model

$$E_2(Y) = \frac{z_1\cdot 120 + z_2\cdot 120 + \cdots}{\binom{52}{5}\cdot 5!}$$

2nd model

Sol#2

A_i : indicator of Ace of i^{th} suit in hand

$$Y = \sum_{i=1}^{4} A_i = \sum_{i=1}^{4} P(\text{Ace}_i \text{ in hand}) = 4 \cdot \frac{5}{52} = \frac{5}{13}$$

x : a fixed card

$B \subseteq$ set of cards $\qquad |B| = 5$

$\boxed{\text{DO}}$ $\rightarrow$ $$P_B(x \in B) = P_{x,B}(x \in B) = P_x(x \in B) = \frac{5}{52}$$

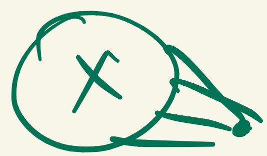
22.25 $4COL \in NPC$

$$3COL \leq_{Karp} 4COL$$

X - graph $\qquad f(X)$ - a graph

<u>Need</u> X is 3-colorable $\iff$ $f(X)$ is 4-colorable

$f(X)$



20.18 $\quad X, Y$ random var.

uncorrel: $E(XY) = E(X)\cdot E(Y)$

not indep: $\neg(\forall x, y \in \mathbb{R})\big(\Pr(X=x \wedge Y=y) = \Pr(X=x)\cdot\Pr(Y=y)\big)$

min $\leftarrow |\Omega|$

$|\Omega| = 3$

$\Omega = [3]$, unif.

Ω	X	Y	XY
1	1	1	1
2	0	0	0
3	-1	1	-1
E	0	×	0

$Y = X^2$

$XY = X$

$\Pr(X=Y=0) = \frac{1}{3}$

$\Pr(X=0)\cdot\Pr(Y=0) = \frac{1}{9}$ ✗

21.39 (b) Cook red. $\not\Rightarrow$ Karp red. under hypothesis

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$$L \prec_{\text{Cook}} M \not\Rightarrow L \prec_{\text{Karp}} M$$

$L = 3COL$ $M = \neg 3COL$ ✓ Cook	if Karp then $NP = coNP$

MAX-3SAT deterministic algorithm

$C_1, \dots, C_m$ 3-clauses

under random assignment

X: # satisfied clauses

$X = \sum_{i=1}^{m} Y_i$ Y_i indicates: i^{th} clause satisfied

$$E(X) = \sum_{i=1}^{m} \underbrace{E(Y_i)} = \sum P(\underbrace{Y_i \text{ } i^{th} \text{ clause satisfied}}_{7/8}) = \frac{7m}{8}$$

$$E(X) = \frac{E(X \mid x_1 = 0) + E(X \mid x_1 = 0)}{2}$$

$N = \Theta(m \log n)$

$N = \Theta((m+n) \log n)$

$(m^2 + n) \log n < N^2$

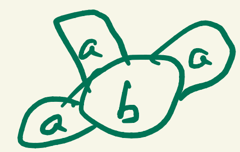
- $\Pr(k\text{-clause satisfied}) = 1 - 2^{-k}$

(a) nontrivial n pairwise indep r.v.'s on $|\Omega| = n+1$

$$b = (a+b)^2$$

$$na + b = 1$$

$$a, b > 0$$



$$P(A_i) = a + b$$

$$P(A_i \cap A_j) = b = (a+b)^2$$

$$na + b = 1$$

$$b = \frac{1}{(n-1)^2} \qquad a = \frac{n-2}{}$$

$$a = \frac{n-2}{(n-1)^2}$$

$$a + b = \frac{1}{n-1}$$

$$a = \frac{1}{n}(1-b) = \frac{1}{n} \cdot \frac{(n-1)^2 - 1}{(n-1)^2} = \frac{(n-2)}{(n-1)^2}$$

18.78 (b) balanced, pairwise indep.

n events $\quad |\Omega| = O(n)$

$< 2n$

$2^{k-1} \le n < 2^k \qquad |\Omega| = 2^k < 2n$ ✓

$\Omega = \mathbb{F}_2^k$ unif.

$\underline{a} \in \mathbb{F}_2^k$, $\underline{a} \ne \underline{0} \qquad V_a = a^\perp = \{\underline{x} \in \mathbb{F}_2^k \mid a \cdot x = 0\}$

subspace of dim $k-1$

$\therefore |V_a| = 2^{k-1} \quad \therefore \Pr(V_a) = \frac{1}{2}$

$V_a \cap V_b \qquad \underline{a \ne b}$

Claim $\llcorner$ has $\underline{\text{codim} = 2}$

$ax = 0$
$bx = 0$

$\underline{a, b \text{ lin indep.}}$ ✓

$\boxed{\text{Do}}$ $\exists$ 3-wise indep, balanced, $O(n)$

NEIL IMMERMAN

ROBERT SZELEPCSÉNYI

NL = coNL

NL: logspace

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trivial $n!$

1980: ISO of graphs of bded $\in P$

group theory EUGENE LUKS 1983

$\exp(\sqrt{n})$

$n^{0.49}$

until 2016 $\exp(\text{poly}(\log n))$ ← quasipolynomial