HONORS COMBINATORICS

2024-03-19

$a \in A$

$A = B \iff (\forall a)(a \in A \leftrightarrow a \in B)$

$|A|$ = cardinality = #elements

$\mathcal{P}(A)$ powerset $= \{B \mid B \subseteq A\}$

If $|A| = n$ then $|\mathcal{P}(A)| = 2^n$

PROOF (no need for Binomial Thm)

binary strings of length n : 2^n

— " — $\xleftrightarrow{\text{bijection}}$ $\mathcal{P}(A)$

member

not member

× • • × • • •

0 1 1 0 1 1 1

binary string

Ω "universe"

$A \subseteq \Omega$

characteristic function
indicator function

$f_A : \Omega \to \{0,1\}$

$$f_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in \Omega \setminus A \end{cases}$$

$f_{A \cup B}$ usually $\neq f_A + f_B$

but

$\boxed{\text{DO}}$ $f_{A \cap B} = f_A \cdot f_B$

Clubtown

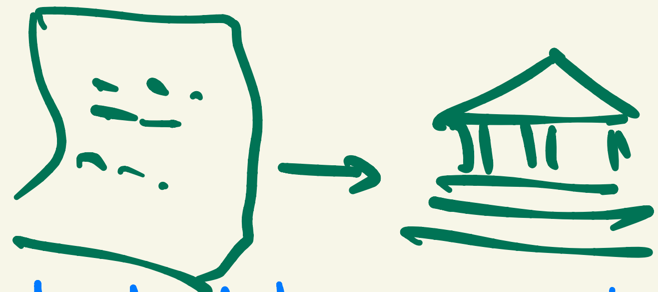
n residents

$$C_1 \ldots C_m \subseteq \Omega \qquad |\Omega| = n$$

(0) $i \neq j \Rightarrow C_i \neq C_j$

max $m = 2^n$

(1) $(\forall i)(|C_i|$ even$)$



$\boxed{\text{DO}}$ if $n \geq 1 \Longrightarrow$ max $m = 2^{n-1}$ # even subsets

Give two proofs

1. "algebra proof": use the BINOMIAL THEOREM
2. "combinatorial proof": "bijective proof": {even subsets} $\leftrightarrow$ {odd subsets}

BOTH PROOFS SHOULD FAIL FOR $n=0$

EVENTOWN

(0) $i \neq j \Rightarrow C_i \neq C_j$

(1) $(\forall i)(|C_i| = \text{even})$

(2) $(\forall i \neq j)(|C_i \cap C_j| = \text{even})$

(2^*) $(\forall i, j)(|C_i \cap C_j| = \text{even})$

floor function

$\lfloor \pi \rfloor = 3$

$\lfloor -\pi \rfloor = -4$

$\max m = 2^{\lfloor n/2 \rfloor}$

EXTREMAL COMBINATORICS

EVENTOWN THEOREM

HW $\max m \geq 2^{\lfloor n/2 \rfloor}$

(need to find Eventown club system with $2^{\lfloor n/2 \rfloor}$ clubs)

CH $m \leq 2^{\lfloor n/2 \rfloor}$

2 weeks: until April 1

(need to prove upper bound for ALL Eventown club systems)

ODDTOWN

$$\left.\begin{array}{l}(1^*)\quad (\forall i)\big(|C_i| = \text{odd}\big)\\ (2)\quad (\forall i\neq j)\big(|C_i\cap C_j| = \text{even}\big)\end{array}\right\}\Rightarrow (O)$$

max m = ?

$C_1, \ldots, C_m \subseteq \Omega \qquad |\Omega| = n$

ODDTOWN THEOREM

(1*) $(\forall i)(|C_i| = \text{odd})$

(2) $(\forall i \neq j)(|C_i \cap C_j| = \text{even})$

$\Rightarrow$ (0)

$\Longrightarrow$ max m $= n$

$\cancel{2^n}$ ① max $m \geq n$

$\cancel{2^{\frac{n-1}{2}}}$ NTS: construct set of n Oddtown clubs

$\cancel{2^{n/3}}$ make $|C_i| = n-1$ works if n : even

simplest: $|C_i| = 1$ "single's clubs"

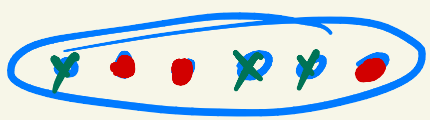
HW $C_1, \dots, C_m \subseteq \Omega \qquad |\Omega| = n$

<u>Oddtown</u> club system:

(1*) $(\forall i)(|C_i| = \text{odd})$

(2) $(\forall i \neq j)(|C_i \cap C_j| = \text{even})$

Let v_i denote the incidence vector of C_i

 $\longrightarrow (0, 1, 1, 0, 0, 1)$

Then $v_1 \dots v_m$ are linearly independent in $\mathbb{R}^n$

easier to prove in $\mathbb{Q}^n$

HW If $u_1, \dots, u_m \in \mathbb{Q}^n$ are lin indep in $\mathbb{Q}^n$ then they are lin. indep. in $\mathbb{R}^n$

$\boxed{\text{CH}}$ # Oddtown systems is $> 2^{\frac{n^2}{8}(1-\varepsilon)}$ $(\forall n > n_\varepsilon)$ [8

$n \to \infty$

asymptotic rates of growth

ASYMPTOTIC EQUALITY

Sequences (a_n), (b_n) $\qquad a_n, b_n \in \mathbb{R}$

are asymptotically equal "$a_n \sim b_n$"

if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1$$

(a_n) is eventually nonzero if

$$(\exists n_0)(\forall n \geq n_0)(a_n \neq 0)$$

$\boxed{\text{DO}}$ If $a_n \sim b_n$ then both (a_n) and (b_n) are eventually nonzero

$\boxed{\text{DO}}$ $\sim$ is an equivalence relation among eventually-nonzero sequences:

(i) $a_n \sim a_n$

(ii) $a_n \sim b_n \implies b_n \sim a_n$

(iii) $\left.\begin{array}{l} a_n \sim b_n \\ b_n \sim c_n \end{array}\right\} \implies a_n \sim c_n$

EX. $17x^5 - 1000x^4 + 3x - 10^6 \sim 17x^5$

Pf Quotient: $1 - \underbrace{\frac{1000}{17}\frac{1}{x}}_{\to 0} + \underbrace{\frac{3}{17}\cdot\frac{1}{x^4}}_{\to 0} - \underbrace{\frac{10^6}{17}\cdot\frac{1}{x^5}}_{\to 0} \to 1$

STIRLING'S FORMULA

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

$\boxed{\text{HW}}$ $\binom{2n}{n} \sim a^n \cdot n^b \cdot c$

a, b, c constant ← find them

PRIME COUNTING FUNCTION

$\pi(x) = \underline{\#\text{ primes } p \le x}$

$\pi(4) = 2$ 2, 3

$\pi(10) = 4$ 2, 3, 5, 7

$\pi(100) = 25$

$\pi(\pi) = 2$

$\pi(-3) = 0$

PRIME NUMBER THEOREM

$$\pi(x) \sim \frac{x}{\ln x}$$

1896 JACQUES HADAMARD
PIERRE de la Vallée Poussin