HONORS COMBINATORICS

2024-03-21

Hypergraph $\mathcal{H} = (V, \mathcal{E})$

V : set of vertices ← vertex

$\mathcal{E}$: " " subsets of V : $\mathcal{E} \subseteq \mathcal{P}(V)$

↑ powerset

Multihypergraph :

$\mathcal{E}$: multiset : each member has a multiplicity

$\mathcal{E} : \mathcal{P}(V) \to \mathbb{N}_0 = \{0, 1, 2, \ldots\}$

Incidence matrix of $\mathcal{H}$

$$V = \{v_1 \dots v_n\}$$

$$\mathcal{E} = \{E_1 \dots E_m\}$$

n : # vertices "order of $\mathcal{H}$"

m : # edges "size of $\mathcal{H}$"

$E_i = E_j$ permitted

$$M(\mathcal{H}) = (m_{ij})_{\substack{i = 1 \dots m \\ j = 1 \dots n}}$$

$$m_{ij} = \begin{cases} 1 & \text{if } v_j \in E_i \\ 0 & \text{if } v_j \notin E_i \end{cases}$$

v_j

E_i ← rows : incidence vectors of edges

1-1 corr. between

multihypergraphs of order n, size m $\longleftrightarrow$ $m \times n$ $(0,1)$ matrices

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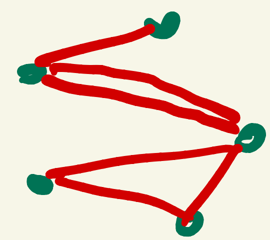
$$\mathrm{rank}(E_i) = |E_i|$$

$$\mathrm{rank}(\mathcal{H}) = \max_{1 \le i \le m} \mathrm{rank}(E_i)$$

$\mathcal{H}$ is r-<u>uniform</u> if $(\forall i)(|E_i| = r)$

2-uniform hypergraph: graph

" multi " : multigraph



$G = (V, E)$

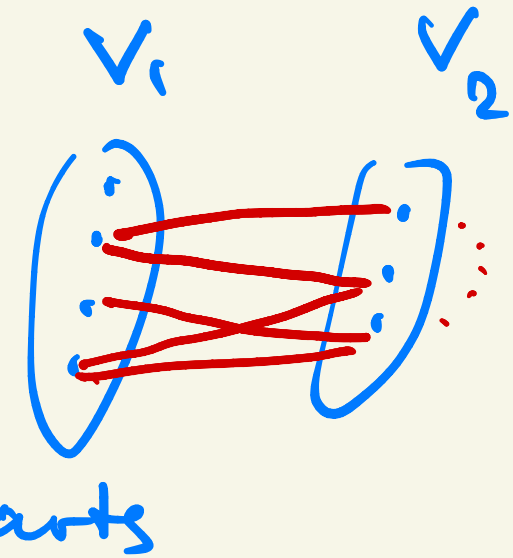
G is a <u>bipartite graph</u> if

$$V = V_1 \sqcup V_2$$

$\uparrow$ disjoint union

$\forall$ edge $e = \{u_1, u_2\}$

$u_1 \in V_1 \quad u_2 \in V_2$



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$\mathcal{H}$ multi-hypergraph $\quad V = \{v_1 \dots v_n\}$

$\mathcal{E} = \{E_1 \dots E_m\}$

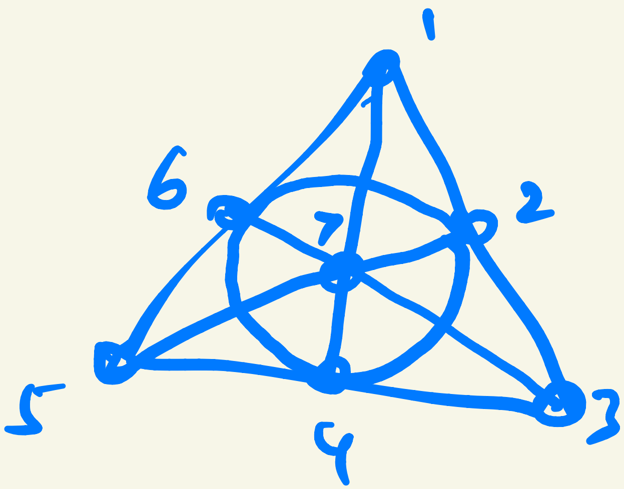
$\downarrow$

bipartite graph $\quad \mathcal{G} = (\overset{V}{\downarrow}{U_1} \sqcup \overset{\mathcal{E}}{\downarrow}{U_2}, F)$

$$\{u_1, u_2\} \in F \iff u_1 \in u_2$$

FANO PLANE

3-uniform hypergraph of order 7
size 7



	1	123
	2.	345
	3	561
	4	174
	5	275
	6	376
l_7	7	246

$\mathcal{E}$

$\{1,2,3\}$

$p_2 \in l_7$

$p_3 \notin l_7$

1	1	1	0	0	0	0
0	0	1	1	1	0	0
1	0	0	0	1	1	0
1			1			1
	1			1		1
		1			1	1
	1		1		1	

← incidence matrix

Dual multi hypergraph

$\mathcal{H}^{tr}$ corr. to transpose of incidence matrix

order of $\mathcal{H}^{tr}$ = size of $\mathcal{H}$

size ... order

$$\mathcal{H}^{tr\,tr} = \mathcal{H}$$

$\boxed{\text{DO}}$ $\text{Fano}^{tr} \cong \text{Fano}$

isomorphic

$\mathcal{H}_1 = (V_1\ \mathcal{E}_1) \qquad \mathcal{H}_2 = (V_2\ \mathcal{E}_2)$

$V_1 = \{v_1 \dots v_n\} \qquad V_2 = \{w_1 \dots w_n\}$

$\mathcal{E}_1 = \{E_1 \dots E_m\} \qquad \mathcal{E}_2 = \{F_1 \dots F_m\}$

Isomorphism $f : \mathcal{H}_1 \to \mathcal{H}_2$

$f = (g, h)$ $\quad$ g: bijection $V_1 \to V_2$

h: " $\quad \mathcal{E}_1 \to E_2$

preserves <u>incidence</u> (membership)

$(\forall i, j)\ (v_j \in E_i \iff g(v_j) \in h(E_i))$

$g(j) \in h(i)$ =

$\mathcal{H}_1$ $\mathcal{H}_2$ are isomorphic

<u>if</u> $\exists$ $f: \mathcal{H}_1 \to \mathcal{H}_2$ isomorphism

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$\boxed{\text{DO}}$ Isomorphism (fact of being isomorphic) is an equiv. rel. on the set of multi hypergraphs

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$\boxed{\text{DO}}$ relation to isomorphism of incidence graphs

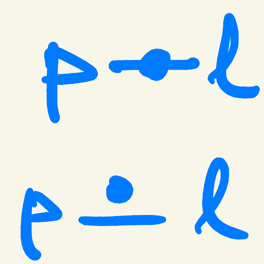
Finite projective plane

$$\mathcal{P} = (P, L, I)$$

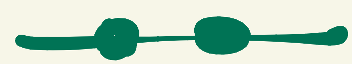
P: set of points (vertices)

L: lines (edges)

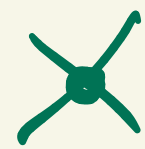
$I \subseteq P \times L$ incidence relation



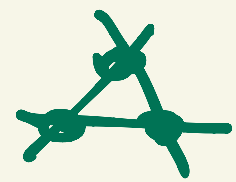
Ax1 $(\forall p_1 \neq p_2 \in P)(\exists!\, \ell \in L)(p_1, p_2 \,{-}\!\bullet\!{-}\, \ell)$



Ax2 $(\forall \ell_1 \neq \ell_2 \in L)(\exists!\, p \in P)(p \,{-}\!\bullet\!{-}\, \ell_1, \ell_2)$

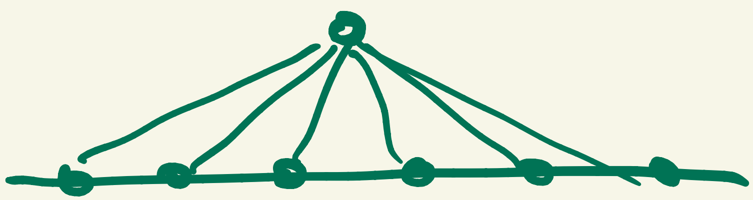


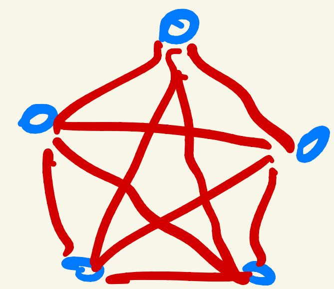
Ax3* $\exists$ triangle: 3 points, not on a line



$$\underbrace{Ax1,\ Ax2,\ Ax3^*}_{\text{possibly degenerate proj. plane}}$$

$\boxed{D}$ $(\forall n \geq 3)(\exists$ with n points$)$

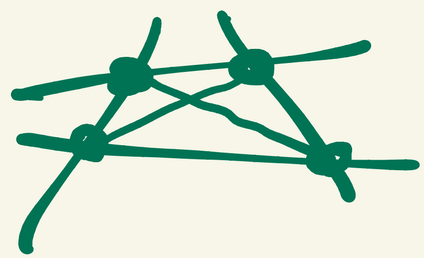




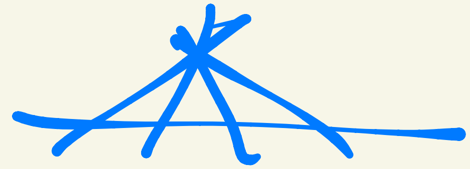
Ax3 $\exists 4$ points, no 3 on a line

$\downarrow$

6 lines



<u>Ch</u> The only degenerate proj. planes are these



<u>Ex</u> If p.p. $\Rightarrow$ uniform
dual is also p.p.

<u># points on a line $-$ 1</u> <u>"order of p.p."</u>

Example: Fano plane: <u>order 2 p.p.</u>

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$\mathcal{H} = (V, \mathcal{E})$

$A \subseteq V$ is <u>independent</u> if

$(\forall E \in \mathcal{E})(E \not\subseteq A)$

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independence number of $\mathcal{H}$

$\alpha(\mathcal{H}) = \max\{|A| \mid A \subseteq V \text{ is indep.}\}$

\alpha

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in SET game: indep. sets $\to$ "capsets"