HONORS COMBINATORICS | 2024-03-26 | 1

Incidence geometry

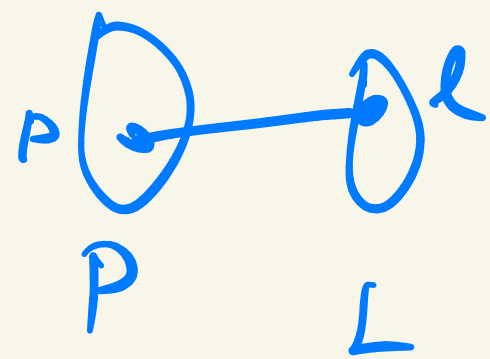
$\mathcal{G} = (P, L, I)$

P, L sets, $I \subseteq P \times L$

P — set of points, L — set of lines, I — incidence relation

$p \multimap l$ means $(p, l) \in I$

Incidence graph



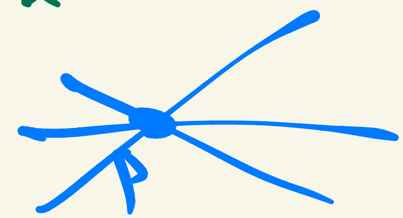
Incidence matrix

n points m lines

$M = (m_{ij})$ $\quad m_{ij} = \begin{cases} 1 & p_j \multimap l_i \\ 0 & \text{o/w} \end{cases}$

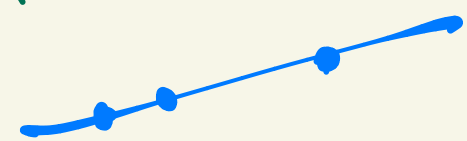
$p \in P$ degree of p

$\deg(p) \doteq \#$ lines ℓ s.t. $p \mathrel{-\!\bullet\!-} \ell$



$\ell \in L$ rank of ℓ

$rk(\ell) = \#$ pts p s.t. $p \mathrel{-\!\bullet\!-} \ell$



$\boxed{\text{DO}}$ $\displaystyle\sum_{p \in P} \deg(p) = \sum_{\ell \in L} rk(\ell)$

<u>DEF</u> $\mathcal{G}$ is d-<u>regular</u> if $(\forall p \in P)(\deg(p) = d)$

<u>regular</u> if $(\exists d)(\mathcal{G}$ is d-regular$)$

r-<u>uniform</u> if $(\forall \ell \in L)(rk(\ell) = r)$

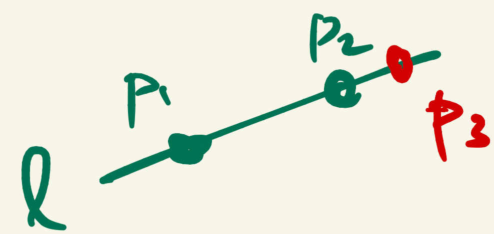
$$\mathcal{G}^{tr} = (L, P, I^{-1})$$

STEINER TRIPLE SYSTEM STS

3-uniform incidence geometry s.t.

Ax $(\forall p_1, p_2 \in P)(p_1 \neq p_2 \Rightarrow (\exists! \, \ell \in L)(p_1, p_2 -\!\!\bullet\!\!- \ell))$

Algebraic view:
binary operation on Ω:
$$f : \Omega \times \Omega \to \Omega$$



If $p_1 \neq p_2$ then $p_1 \circ p_2 := p_3$ s.t. $\overbrace{\{p_1, p_2, p_3\}} \in L$

$p \circ p := p$

NOTE: $p_1 \circ p_2 = p_2 \circ p_1$

$$p_1 \circ p_2 = p_3 \implies p_1 \circ p_3 = p_2$$

STS

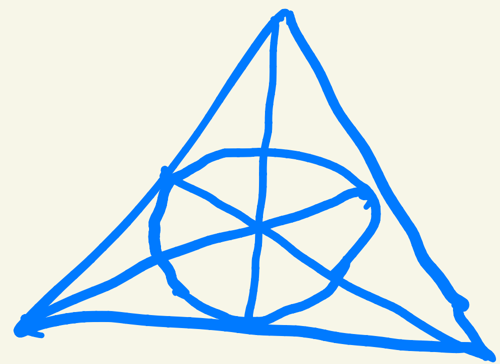
$|P| = 1 \quad L = \emptyset$

$|P| = 3 \quad |L| = 1$

?

FANO plane

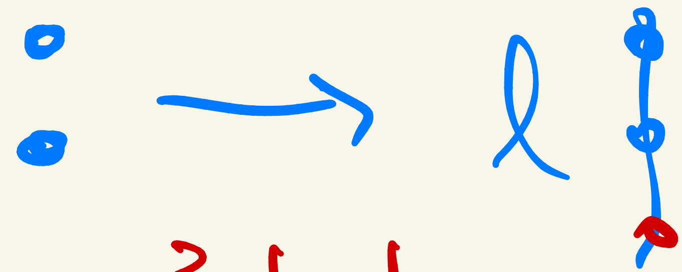
$|P| = |L| = 7$



n points

$\binom{n}{2}$ pairs of points

$$m = |L| = \frac{\binom{n}{2}}{3} = \frac{n(n-1)}{6}$$



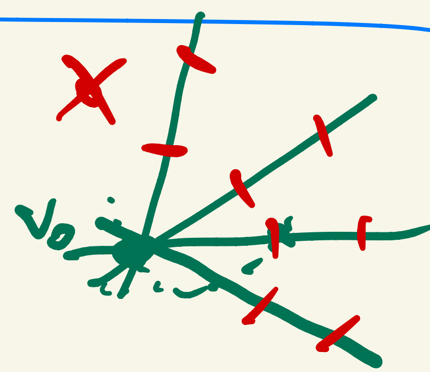
3-to-1 corr.

thm $6 \mid n(n-1)$

divisor

---

thm n must be odd



$\therefore$ $n \neq 5$

$$\left.\begin{array}{l} 2 \nmid n \\ 3\ \cancel{6} \mid n(n-1) \end{array}\right\}$$

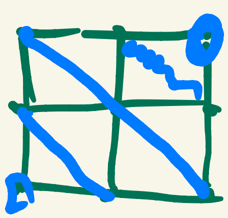
$\boxed{\text{DO}}$ $\iff$ $n \equiv 1$ or $3 \mod 6$

1, 3, 7 ✓ 9 ?

$$n = 9 \implies m = \frac{(n-1)n}{6} = \underline{\underline{12}}$$

2D-SET card game

two attributes



$P = \mathbb{F}_3^2$

L : lines in this plane

affine plane over $\mathbb{F}_3$ field of order 3

$\mathcal{A} = (A, \circ)$ $\quad$ $\circ\,,\ \ast$: binary operations

$\mathcal{B} = (B, \ast)$

Direct product:

$$\mathcal{A} \times \mathcal{B} = (A \times B, \square)$$

$$(a_1, b_1) \,\square\, (a_2, b_2) = (a_1 \circ a_2, b_1 \ast b_2)$$

If $\mathcal{A}, \mathcal{B}$ commutative $\Longrightarrow$ $\mathcal{A} \times \mathcal{B}$ also

$\boxed{\text{DO}}$ If $\mathcal{A}, \mathcal{B}$ satisfy $a_1 \circ a_2 = a_3 \Longrightarrow a_1 \circ a_3 = a_2$

then $\mathcal{A} \times \mathcal{B}$ also

$\boxed{\text{DO}}$ These three axioms define STS

$a \circ a = a$ idempotent also inherited

COROLLARY If $\exists$ STS w k points

" ℓ "

$\Longrightarrow \exists$ STS w $k \cdot \ell$ points

$\mathcal{A} = (A, \circ)$ $\quad \circ$ binary op.

$B \subseteq A$

$\mathcal{B} = (B, \circ)$ is a subalgebra

if B is closed under $\circ$

HW If $\mathcal{A}$ STS, $\mathcal{B}$ proper subSTS

then $|B| \leq \frac{n-1}{2}$ $\qquad n = |A|$

THM STS exists w n points $\iff n \equiv 1$ or $3 \bmod 6$

$\Longrightarrow$ ✓

$\Longleftarrow$?

Starter cases

+ gluing

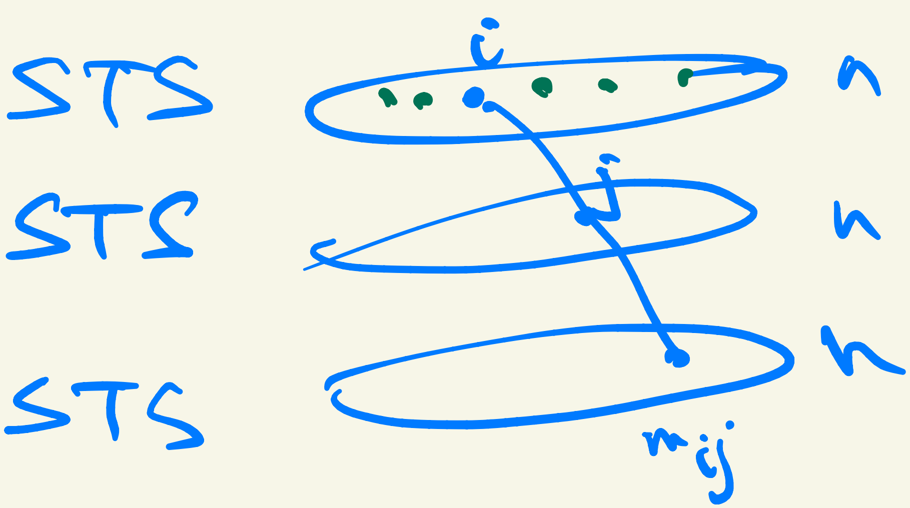
LATIN SQUARE of order n

$n \times n$ matrix filled w/ $\{1, \dots, n\} = [n]$ s.t.

$\forall$ row, $\forall$ column has exactly one of each $i \in [n]$

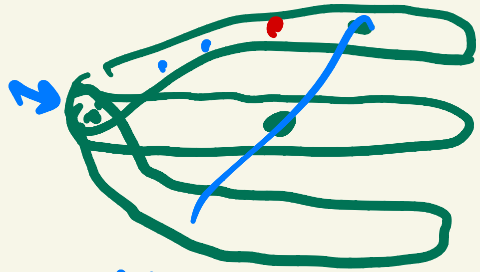
$$\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{matrix}$$

$$\begin{matrix} 1 & 2 & \dots & n \\ 2 & 3 & \dots n & 1 \end{matrix}$$



glue: Latin square

STS
STS
STS
order n

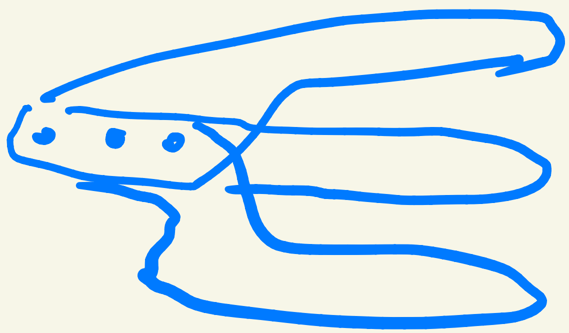


m_{ij}

$(n-1)\times(n-1)$ Latin Sq.

$\rightarrow n' = 3n-2$

cf
$n=13$
elegant



$n' = 3n-6$

Fano →

$n' = 3n-14$