COMBINATORICS PROBLEM SESSION

2024-03-29

1

Oddtown $|\Omega| = n$ $C_i \subseteq \Omega$

(i) $|C_i|$ odd

(ii) $i \neq j \Rightarrow |C_i \cap C_j| = \text{even}$

$\Longrightarrow$ $v_1 \cdots v_m$ lin independent over $\mathbb{Q}, \mathbb{R}, \mathbb{F}_2$

v_i: indicator vector of C_i

$v_i \cdot v_j = |C_i \cap C_j|$

2

$$v_i \cdot v_j = |C_i \cap C_j| = \begin{cases} \text{odd} & \text{if } i=j \\ \text{even} & \text{if } i \neq j \end{cases}$$

Over $\mathbb{F}_2$

$$v_i \cdot v_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

NTS $\forall a_1 \cdots a_m$

if $w = \sum a_i v_i = 0$ then $(\forall i)(a_i = 0)$

$$0 = v_k \cdot w = \sum a_i \cdot \underbrace{v_k \cdot v_i}_{\neq 0 \text{ only for } i=k} = a_k \quad \checkmark$$

Over $\mathbb{Q}$

$a_i \in \mathbb{Q}$

WLOG $a_i \in \mathbb{Z}$ | $\sum a_i v_i = 0$ ($\exists$

(multiply by lcm denominators)

for →←

assume, not all coeff's are zero

WLOG $\gcd(a_1 \dots a_m) = 1$

$$0 = v_k \cdot \sum a_i \cdot \underbrace{v_k \cdot v_i} = \text{even} \cdot \sum_{i \neq k} a_i + \text{odd} \cdot a_k$$

$\Longrightarrow a_k$ even $\forall k$

→← $\gcd = 1$

Recall $\gcd(0, 0, \dots, 0) = \underline{\underline{0}}$

Find $\underline{a}, \underline{b}, \underline{c} \in \{0,1\}^k$. k small

lin indep over $\mathbb{Q}$, $\mathbb{R}$

but dep over $\mathbb{F}_2$

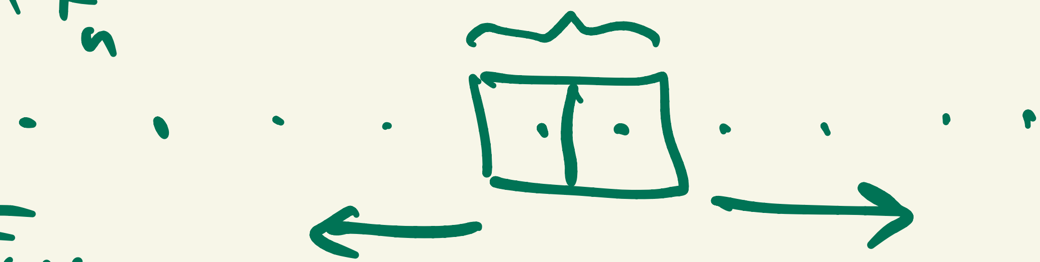
a	0	1	1
b	1	0	1
c	1	1	0
s	2	2	2

$s-2a$	1	0	0
$s-2b$	0	1	0
$s-2c$	0	0	1

$$\det\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = 2$$

$(F_n \bmod m)$ periodic with period $\leq m^2 - 1$

$$F_{n+2} = F_{n+1} + F_n$$



$$F_n = F_{n+2} - F_{n+1}$$

window repeats by PHP

view

$m = 3$ per 8

n^{th} prime $\qquad$ prime counting fct

$$p_n \sim n \cdot \ln n \iff \text{PNT} \quad \pi(x) \sim \frac{x}{\ln x}$$

ASSN PNT $\qquad$ Desired conclusion $\quad p_n \sim n \cdot \ln n$

DC

$$\pi(p_n) = n$$

$$n = \pi(p_n) \sim \frac{p_n}{\ln p_n}$$

NTS

$\ln p_n \sim \ln n$

$$p_n \sim n \cdot \ln p_n$$

$$\underline{\ln p_n} \sim \ln n + \underline{\ln \ln p_n}$$

$$1 = \lim \frac{\ln n}{\ln p_n} + \frac{\ln \ln p_n}{\ln p_n}$$

$\downarrow$ 0

$$\therefore \ln p_n \sim \ln n$$

ASSN $p_n \sim n \cdot \ln n$ NTS $\pi(x) \sim \dfrac{x}{\ln x}$

for $x = p_n$

DC

$$\boxed{n \sim \frac{p_n}{\ln p_n}}$$

$$\begin{cases} p_n \sim n \ln n \\ \ln p_n \sim \underline{\ln n + \ln \ln n} \sim \underline{\ln n} \end{cases}$$

$p_n \sim n \cdot \ln p_n$

$p_n \le x < p_{n+1}$ $\pi(x) = n$

$n \ln n \sim p_n \le x < p_{n+1} \sim (n+1) \ln (n+1) \sim n \ln n$

SQUEEZE $x \sim n \ln x \sim (n+1) \ln (n+1)$

$x \ln x$

NOT "eventually nonzero"

"a_n is zero infinitely often"

0 1 0 1 0 . . .

"Some"

$p_n \cdots n \ln n$

$\pi(p_n) = n$

$p_n \leq x < p_{n+1}$

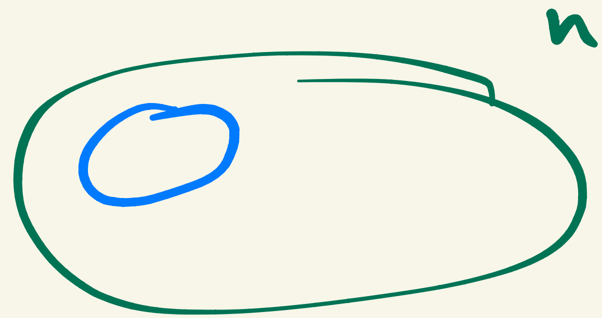
$x := n \ln n$

$\pi(x) \overset{?}{\sim} n$

Smallest maximal Oddtown system

1 if n odd

2 if n even ~~odd odd~~



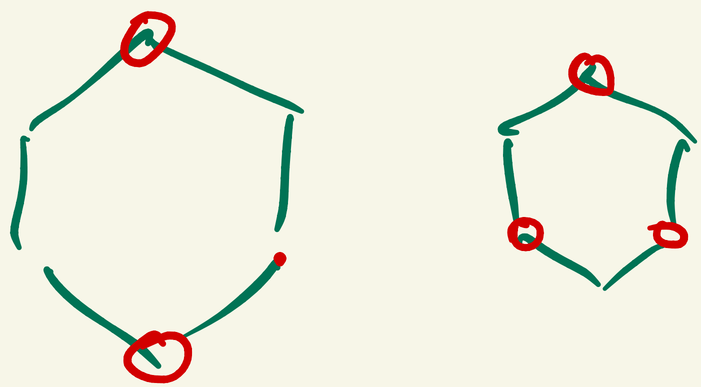
Claim $\{[n]\}$ is maximal

Pf Assume for $\rightarrow\leftarrow$ $C_2 \neq [n]$

but then $|C_2| = \text{odd}$

$|C_2| = |C_2 \cap [n]| = \text{even}$

in what context is maximal always maximum



1.87b

$\varepsilon > 0$ fixed

$a_n \sim b_n$

$\Longrightarrow \ln a_n \sim \ln b_n$

for $n \geq n_0$

$a_n \geq 1 + \varepsilon \Longrightarrow b_n \geq 1 + \frac{\varepsilon}{2}$ $n \geq n_1$

Why is $a_n, b_n > 1$ not sufficient

$a_n = 1 + \frac{1}{n}$ $b_n = 1 + \frac{1}{n^2}$

$\ln(1+\frac{1}{n}) \sim \frac{1}{n}$

$\ln(1+\frac{1}{n^2}) \sim \frac{1}{n^2}$

$\lim\limits_{x \to 0} \frac{\ln(1+x)}{x} = 1$

$a_n = e^{1/n}$

$b_n = e^{1/n^2}$

1

$\ln a_n = \frac{1}{n}$

$\ln b_n = \frac{1}{n^2}$

ASSN $a_n, b_n \geq 1+\varepsilon$

$a_n \sim b_n$ | DC $\ln a_n \sim \ln b_n$

$$\frac{a_n}{b_n} \to 1$$

$$\ln\left(\frac{a_n}{b_n}\right) \to 0$$ b/c $\ln$ is continuous at 1

$$\ln a_n - \ln b_n \to 0$$

$$\left|\frac{\ln a_n}{\ln b_n} - 1\right| = \left|\frac{\ln a_n - \ln b_n}{\ln b_n}\right| \leq \left|\frac{\ln a_n - \ln b_n}{\underbrace{\ln(1+\varepsilon)}_{=\delta > 0}}\right| = \frac{1}{\delta}\underbrace{|\ln a_n - \ln b_n|}_{\downarrow\, 0} \to 0$$

(13

$$\binom{n}{5} \sim c \cdot n^5 \qquad c = \frac{1}{5!} = \frac{1}{120}$$

$$\frac{n(n-1)\cdots(n-4)}{5! \cdot n^5} \sim c$$

$$\frac{1}{5!}\,\frac{n}{n}\cdot\frac{n-1}{n}\cdots\frac{n-4}{n} \to c$$

$$\underbrace{\binom{n}{5} \sim \frac{n^5}{120}}$$

$$\underbrace{\underbrace{\frac{n}{n}}_{\downarrow\,1}\cdots\underbrace{\left(1-\frac{4}{n}\right)}_{\downarrow\,1}}_{1}$$

$$\frac{1}{5!} \sim \downarrow \to c$$

NO!

$$\binom{n}{5} = \frac{n!}{5!\,(n-5)!}$$

YES

$$\binom{2n}{n} \sim \frac{1}{\sqrt{\pi}}\,\frac{4^n}{\sqrt{n}}$$

$\ln(n!) \sim n\ln n$ w/o Stirling

$$\left(\frac{n}{e}\right)^n < n! < n^n \quad \checkmark$$

$$n(\ln n - 1) < \ln(n!) < n\ln n$$

Squeeze

$$n! > \left(\frac{n}{e}\right)^n$$

because

if $x \geq 0$

$$e^x = \sum \frac{x^k}{k!} > \frac{x^n}{n!}$$

$$e^n > \frac{n^n}{n!}$$

Eventown tight

need to show

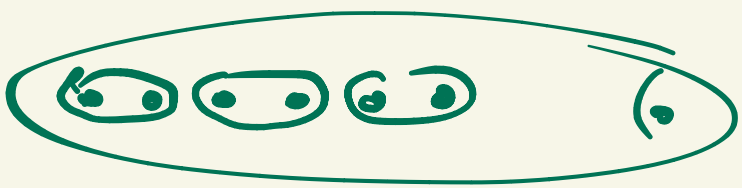
$m \le 2^{\lfloor n/2 \rfloor}$

#clubs

tight:

$\exists\ C_1 \ldots C_m$ Eventown system with $m = 2^{\lfloor n/2 \rfloor}$

"married couples" solution



$\lfloor \frac{n}{2} \rfloor$ couples go together

$\downarrow$

$2^{\lfloor \frac{n}{2} \rfloor}$ clubs