HONORS COMBINATORICS

2024-04-02

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$\mathbb{F}$ field

$\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{F}_p \leftarrow$ mod p residue classes

Scalars

set, +, x

4 arithm. operations can be performed, except no division by 0

usual identities hold

$\mathbb{F}^d$: d-dim vectors $\left\{ \begin{bmatrix} a_1 \\ \vdots \\ a_d \end{bmatrix} : a_i \in \mathbb{F} \right\}$

vector $[a_1, \dots, a_d]^{tr}$

linear combination $\sum_{i=1}^{n} \alpha_i v_i$

$\alpha_i \in \mathbb{F}$

$v_i \in \mathbb{F}^n$

linear

subspace: $U \subseteq \mathbb{F}^n$

closed under linear combinations

COR $\underline{0} \in U$

$v_1 \dots v_k$ are linearly independent

if $(\forall \alpha_1 \dots \alpha_k \in \mathbb{F})(\sum \alpha_i v_i = \underline{0} \Rightarrow (\forall i)(\alpha_i = 0))$

$U := \mathrm{Span}(v_1 \dots v_k) = \{$all lin combs of the $v_i\}$

$\boxed{\text{DO}}$ ↑ a subspace $U \leq \mathbb{F}^n$

$\dim U =$ max size of a lin. indep. subset

1st MIRACLE of LIN ALG

$\boxed{\text{FACT}}$ Let $S \subseteq \mathbb{F}^n$

Every maximal lin indep set in S is maximum ← basis

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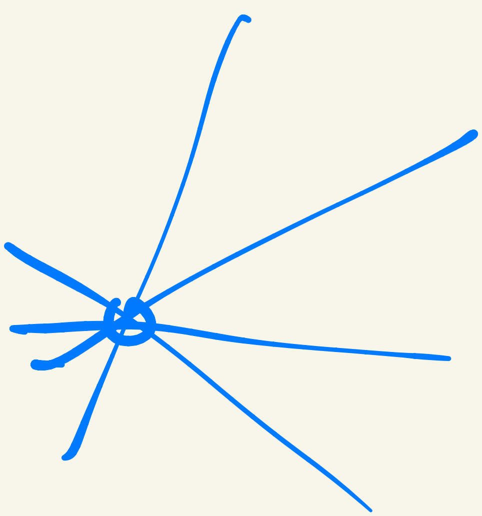
size of this max indep set: rank of S

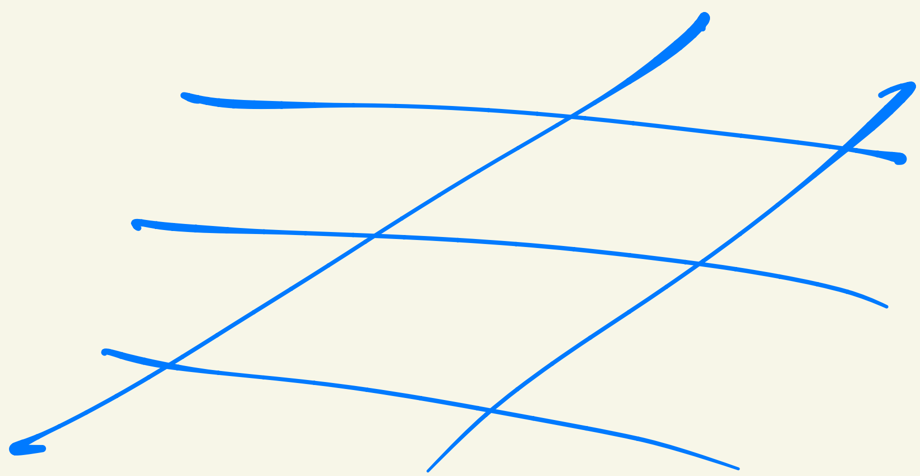
if $S \leq \mathbb{F}^n$ then $\mathrm{rk}(S)$ is called <u>dimension</u>

Greedy algorithm always finds basis

So far: $\mathbb{F}^n$ viewed a linear space
(vector -"-)

$\mathbb{F}^n$ viewed as an affine space





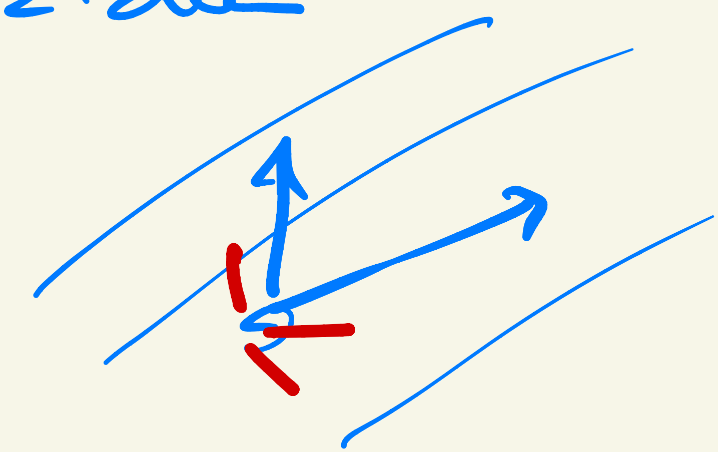
affine combination of

$v_0, v_1, \ldots, v_k$

$$\sum_{i=0}^{k} \alpha_i v_i \qquad \text{s.t.} \quad \underline{\sum_{i=1}^{k} \alpha_i = 1}$$

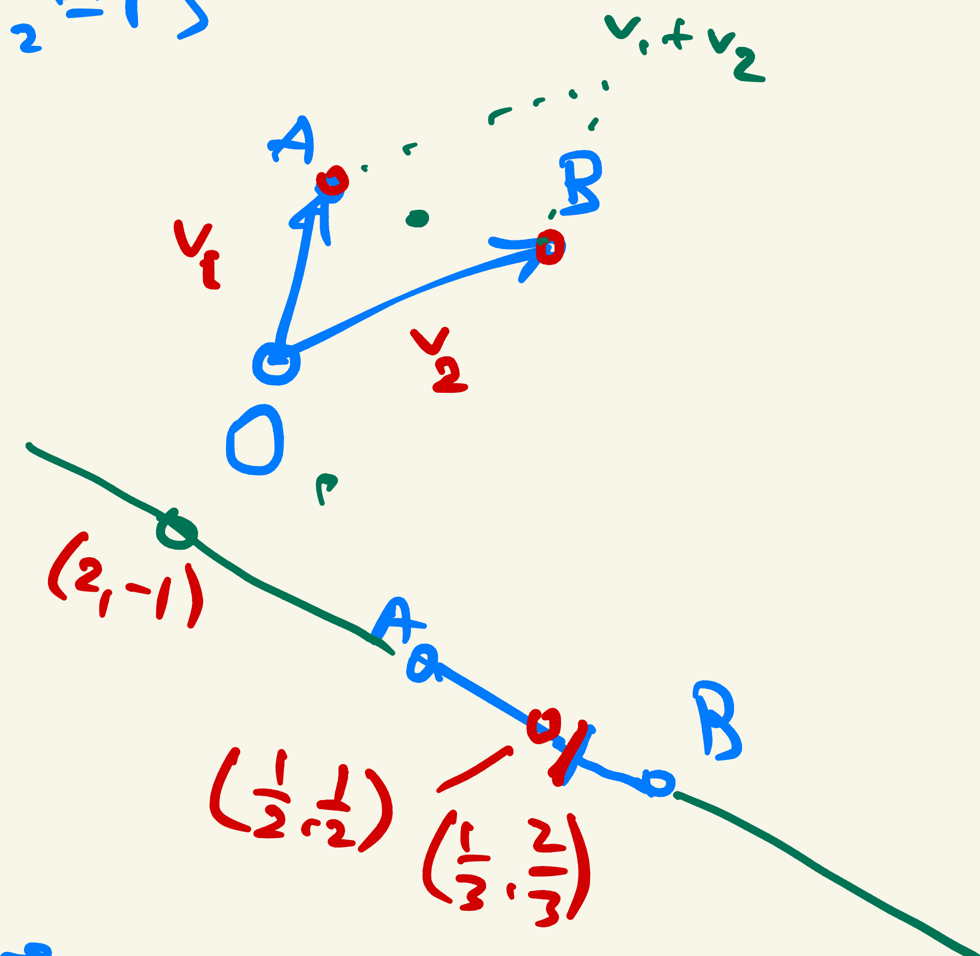
Linear: what are the 2-dim subspaces of $\mathbb{R}^3$

plane through origin



$\{$ all affine comb's of $v_1, v_2 \}$ = line $\overline{AB}$

$\{ \alpha_1 v_1 + \alpha_2 v_2 \mid \alpha_1 + \alpha_2 = 1 \}$



$\alpha_1 = 0 \quad \alpha_2 = 1$

$1 \quad 0$

$\alpha_1 = \alpha_2 = \frac{1}{2}$

$\alpha_1 = \frac{1}{3} \quad \alpha_2 = \frac{2}{3}$

$\alpha_1 = 2 \quad \alpha_2 = -1$

$\boxed{\text{DO}}$ Every point on $\overline{AB}$ line corresp. to a unique aff. comb of v_1, v_2

Affine subspace: $W \subseteq \mathbb{F}^n$
closed under affine comb's, $W \neq \emptyset$

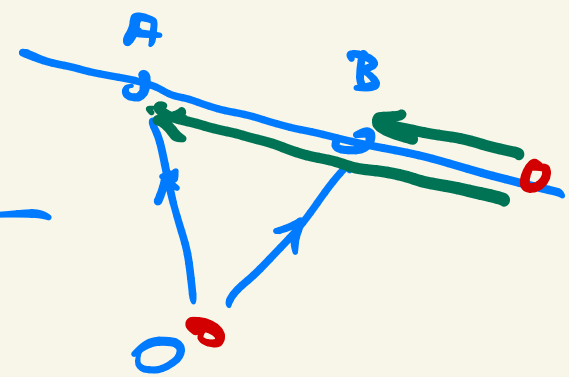
$\operatorname{aff}(S) = \{\text{all aff. comb's of } S\}$
affine closure of $S \subseteq \mathbb{F}^n$

$\operatorname{aff}(\emptyset) = \emptyset$

$\operatorname{aff}(S) \begin{cases} \text{empty} \\ \text{aff subspace} \end{cases}$

DEF $v_0, \dots, v_k$ are <u>affine indep</u> if

$$(\forall \alpha_0 \dots \alpha_k)\left(\left[\begin{array}{l}\text{if } \sum \alpha_i v_i = 0 \\ \text{and} \\ \sum \alpha_i = 0\end{array}\right] \text{ then } (\forall i)(\alpha_i = 0)\right)$$



$U \leq_{\text{aff}} \mathbb{F}^n$

$\dim U = (\max \# \text{ aff indep vectors in } U) - 1$

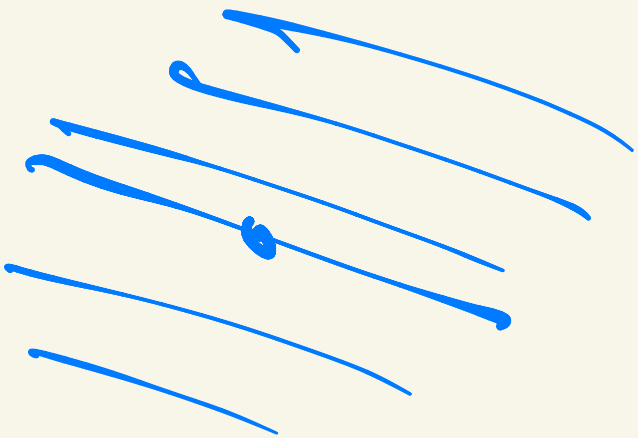
So if $v_0, \dots, v_k$ are aff indep then $\dim(\text{aff}(v_0 \dots v_k)) = k$

$\boxed{\text{DO}}$ Let $U \leq_{lin} \mathbb{F}^n$

$U+b \leq_{aff} \mathbb{F}^n$

and

$\dim(U) = \dim_{aff}(U+b)$

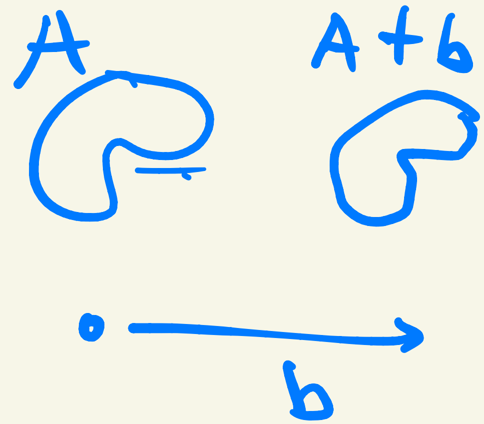


DEF $A \subseteq \mathbb{F}^n$

$b \in \mathbb{F}^n$

$A+b = \{a+b \mid a \in A\}$

shifting A by $\underline{b}$



$A, B \subseteq \mathbb{F}^n$

Sumset $A+B = \{a+b \mid a \in A, b \in B\}$

$\boxed{\text{DO}}$ $W \le_{aff} \mathbb{F}^n$ $\quad w \in W$

$\Longrightarrow$ $W - w \le_{lin} \mathbb{F}^n$ $\quad$ same dim

$\boxed{\text{DO}}$ An aff. subspace W is a lin subspace $\Longleftrightarrow$ $0 \in W$

If $\mathbb{F} = \mathbb{F}_q$ — field of order q, q prime power

and $U \le_{lin} \mathbb{F}_q^n$
$\dim(U) = d$ $\Big\}$ $\Longrightarrow |U| = q^d$

$\Longrightarrow$ same if $U \le_{aff} \mathbb{F}^n$

$\dim_{aff}(U) = d$

Proof: basis $v_1 \dots v_d$

$\forall u \in U \ \exists! \alpha_i$

$u = \sum \alpha_i v_i$

$\mathbb{F}_3^d$, lines

$\mathbb{F}_3 = \{0, 1, 2\}$

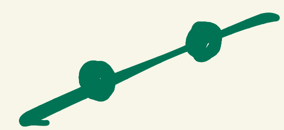
$\uparrow$

1-dim aff subspace T

$|T| = 3^1 = 3$

$\boxed{\text{DO}}$ in $\mathbb{F}^d$ ($\forall$ field)

$(\forall x, y \in \mathbb{F}^d)(x \neq y \Rightarrow \exists!$ line through $x, y)$



$\mathbb{F}_3$: STS

d-dim SET card game

$U \leq \mathbb{F}^n$

$$\operatorname{codim}(U) = \min\{k \mid \overset{\exists v_1 \dots v_k}{\operatorname{span}(U, v_1, \dots, v_k) = \mathbb{F}^n}\}$$

$$= n - \dim U$$