HONORS COMBINATORICS

2024-04-04

$$\mathcal{H} = (V, \mathcal{E}) \qquad \mathcal{E} = \{E_1 \ldots E_m\}$$

$W \subseteq V$ is <u>independent</u> if $(\forall i)(E_i \not\subseteq W)$

Independence number

$$\alpha(\mathcal{H}) = \max\{|W| \mid W \text{ indep.}\}$$

\alpha

If $(\exists i)(E_i = \emptyset)$ then

there are no indep. sets

$\alpha <$ undefined

$\underline{\underline{-\infty}}$

If W indep set and $T \subseteq W$

then T indep.

$\therefore$ If $\emptyset \notin \mathcal{E}$ then $\emptyset$ indep.

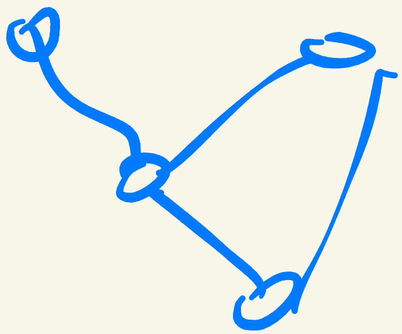
Graph: 2-uniform hypergraph

NP-hard to dist.

$\alpha < n^{\varepsilon}$

$\alpha > n^{1-\varepsilon}$

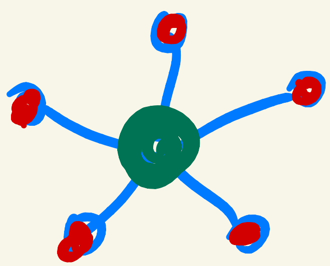
If poly time alg. tells these apart $\Rightarrow P = NP$



large α

but $\exists$ small maximal indep. set

Star



$\alpha = n-1$

$\exists$ maximal indep set of size 1

(4

$\mathrm{SET}_d = (D_d, L_d)$ affine lines

deck of size 3^d : $\mathbb{F}_3^d$

$\boxed{\text{EX}}$ $\underline{a}, \underline{b}, \underline{c} \in \mathbb{F}_3^d$

then

$\underline{a}+\underline{b}+\underline{c}=0 \iff$ either $\underline{a}=\underline{b}=\underline{c}$

or $\{\underline{a}, \underline{b}, \underline{c}\}$ aff. line

$\alpha(\mathrm{SET}_d) \geq 2^d$

$\leq 3^d$

$$2 \le \alpha(SET_d)^{1/d} \le 3$$

$$21^{1/4} \le \quad\quad\quad\quad \overset{?}{<} 2.9999$$

$$\alpha_d := \alpha(SET_d)$$

$\boxed{\text{EX}}$

$$\alpha_{k+\ell} \ge \alpha_k \cdot \alpha_\ell$$

supermultiplicative

Event: $A \subseteq \Omega$

EXAMPE: <u>full house</u>

$P : \Omega \to \mathbb{R}$

$\downarrow$

extend it to

$P : \underbrace{\mathcal{P}(\Omega)}_{\text{powerset}} \to \mathbb{R}$

$\mathcal{P}(\Omega) = \{A \mid A \subseteq \Omega\}$

$\forall A \subseteq \Omega$

DEF $P(A) = \sum_{a \in A} P(a)$

$(\forall A \subseteq \Omega)(0 \leq P(A) \leq 1)$

elementary event: $\{a\}$ $\quad a \in \Omega$

$$P(\{a\}) = P(a)$$

DEF Trivial event: $P \left\langle \begin{matrix} 0 \\ 1 \end{matrix} \right.$

EX $\emptyset, \Omega$

UNION BOUND $\quad P(A_1 \cup \dots \cup A_k) \le \sum_{i=1}^{k} P(A_i)$ $\quad \boxed{\text{DO}}$

when is it <u>equal</u>

$$\iff (\forall i \neq j)(P(A_i \cap A_j) = 0)$$

DEF $A, B \subseteq \Omega$ are independent

if $P(A \cap B) = P(A) \cdot P(B)$

A, B positively correlated if

$$P(A \cap B) > P(A) \cdot P(B)$$

negatively

$$<$$

$\boxed{\text{EX}}$ $([n], \text{unif})$ pick x

A_n: "$2 \mid x$"

B_n: "$3 \mid x$"

are A_n, B_n pos corr / indep / neg. corr } when

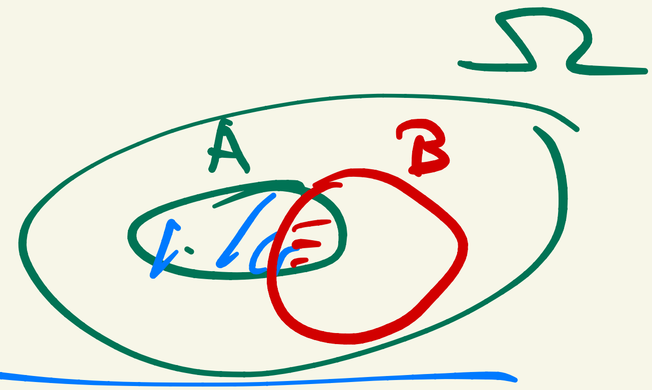
if $6 \mid n$ → indep

Conditional prob

$P(B) \neq 0$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

prob of A given B



If $P(B) \neq 0$

then

A, B — pos / ind / neg $\iff P(A \mid B) \; > \, / = / < \; P(A)$

$\boxed{\text{DO}}$ If B is trivial event then $(\forall A)(A, B \text{ indep})$

DEF A, B, C indep if

they are <u>pairwise</u> indep

$$\left.\begin{array}{l}\left.\begin{array}{ll} A, B & \text{ind} \\ A, C & '' \\ B, C & '' \end{array}\right\} \\ P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C) \quad \} \end{array}\right] \text{DEF}$$

$\boxed{\text{EX}}$ Find pr. space, 3 events s.t. pairwise but not <u>fully</u> indep., min $\leftarrow |\Omega|$

DEF. $A_1 \dots A_n$ are indep. if

$$(\forall I \subseteq [k])\left(P\left(\bigcap_{i\in I} A_i\right) = \prod_{i\in I} P(A_i)\right)$$

$\boxed{\text{EX}}$ If $A_1 \dots A_k$ indep. nontrivial

$$\Longrightarrow |\Omega| \geq 2^k$$

$\boxed{\text{EX}}$ Find k pairwise indep nontriv events in prob. space with $|\Omega| = k+1$

$\boxed{\text{CH}}$ If $\exists k$ pwise ind. nontr. events then $|\Omega| \geq k+1$