COMBINATORICS PROBLEM SESSION

2024-04-05

1

Obs If $\exists A \xrightarrow{inj} B \Rightarrow |A| \le |B|$

1.115 $A_1 \dots A_m \subseteq [n]$

$(\forall i)\quad |A_i| \ge 6$

$(\forall i \neq j)\quad |A_i \cap A_j| \le 5$

$\Longrightarrow \quad m \le \binom{n}{6}$

tight: takes $\binom{[n]}{6}$

$A_i' \subseteq A_i \qquad |A_i'| = 6$

$A_i \mapsto A_i'$ <u>injective</u> b/c if $A_i' = A_j'$

$\Rightarrow |A_i \cap A_j| \ge 6$

$\Rightarrow i = j$

$[m] \xrightarrow[inj]{} \binom{n}{6}$

$\therefore m \le \binom{n}{6}$

1.161

A rational $m \times n$ matrix

$$\Longrightarrow \mathrm{rk}_{\mathbb{Q}}(A) = \mathrm{rk}_{\mathbb{R}}(A)$$

$$\mathrm{rk}_{\mathbb{F}}(A) = \max\{ r \mid A \text{ has an } r \times r \text{ nonsingular submatrix } B \}$$

$\downarrow$

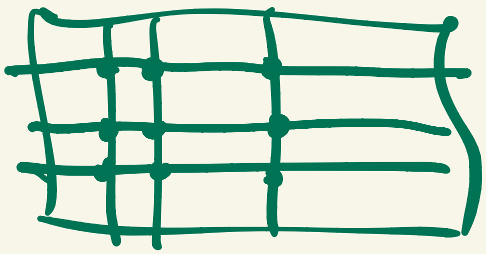
$$\boxed{\det B \neq 0}$$

rows lin indep

col's "

$\exists A^{-1}$

$A\underline{x} = \underline{0}$ has no nontriv sol'n



$$\det B = \sum_{\sigma} \pm \prod b_{i,\sigma(i)}$$

σ perm$[r]$

$$S(n,k) = \sum_{i=0}^{\infty} \binom{n}{ik} = \sum_{i=0}^{\lfloor \frac{n}{k} \rfloor} \binom{n}{ik}$$

(3

$$S(n,1) = \sum_{i=0}^{n} \binom{n}{i} = 2^n$$

$$\binom{n}{k} = \left| \binom{[n]}{k} \right|$$

$$S(n,2) = \sum \binom{n}{2k} = 2^{n-1} = \frac{2^n}{2}$$

Combinat. pf: $\{\text{even subsets}\} \longleftrightarrow \{\text{odd subsets}\}$

bijection

flip membership status of special element

Algebra pf: $(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$

$$2^n = (1+1)^n = \sum \binom{n}{i}$$

$$0^n = (1-1)^n = \sum (-1)^i \binom{n}{i}$$

$$0^n = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{o/w} \end{cases}$$

$$2^n + 0^n = 2 \cdot \sum \binom{n}{2i}$$

$$2^{n-1} = \sum \quad '' \quad \text{if } n \geq 1 \quad \Big| \quad n=0 \quad \text{sum} = 1$$

$$S(n,3) = \frac{2^n}{3} \; ?$$ NO never an integer

$$\left| S(n,3) - \frac{2^n}{3} \right| < 1$$

$$\boxed{\begin{gathered}\omega^3 - 1 = \\ \underline{(\omega-1)}(\omega^2+\omega+1)\end{gathered}}$$

Pf
$$(1+1)^n = 2^n$$
$$(1+\omega)^n = (-\omega^2)^n \quad \text{abs v.} = 1$$
$$(1+\omega^2)^n = \quad \text{—''—}$$

$$\omega^3 = 1 \quad \omega \neq 1$$

$$\boxed{\omega^2 + \omega + 1 = 0}$$

$$\omega + 1 = -\omega^2$$

$$3 \cdot \sum_{i=0}^{\infty} \binom{n}{3i} = 2^n + a + b$$

$$1 = |\omega^3| = |\omega|^3$$

$$|\omega| = 1$$

$$|a|, |b| = 1$$

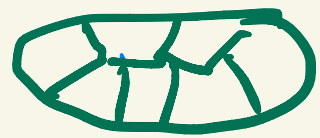
$$\left| \sum \binom{n}{3i} - \frac{2^n}{3} \right| = \left| \frac{a+b}{3} \right| \leq \frac{|a|+|b|}{3} = \frac{2}{3}$$

$H(n,r)$ = # non-isomorphic r-unif hypergraphs on n vertices

$$\frac{2^{\binom{n}{r}}}{n!} \le H(n,r) \le 2^{\binom{n}{r}}$$

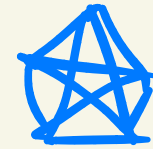
WLOG $V = [n]$

$H(n,r)$: # equivalence classes ← by isomorphism of r-unif hyp on $[n]$



Size of each eq. class $\le n!$ ✓

" of eq. class of hypergraph $\mathcal{H}$

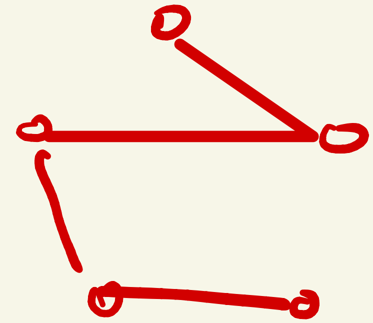


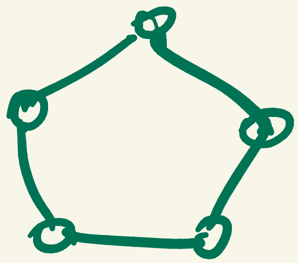
$r = 2$ graphs

$$H(n,r) = n! \sum_{\mathcal{H}} \frac{1}{|\mathrm{Aut}(\mathcal{H})|} = 2^{\binom{n}{r}}$$

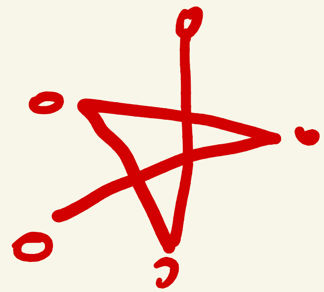
Erep

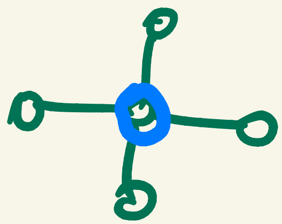
$60 = \frac{120}{2}$



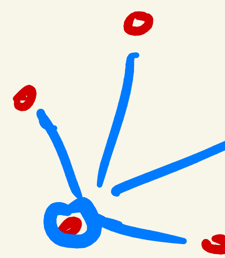


$12 = \frac{120}{10}$





$5 = \frac{120}{24}$

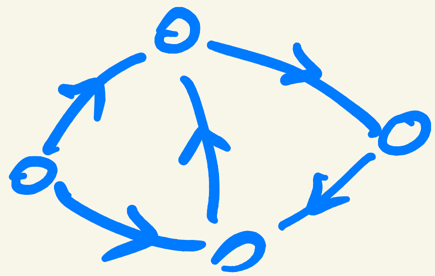


copies of $\mathcal{H}$

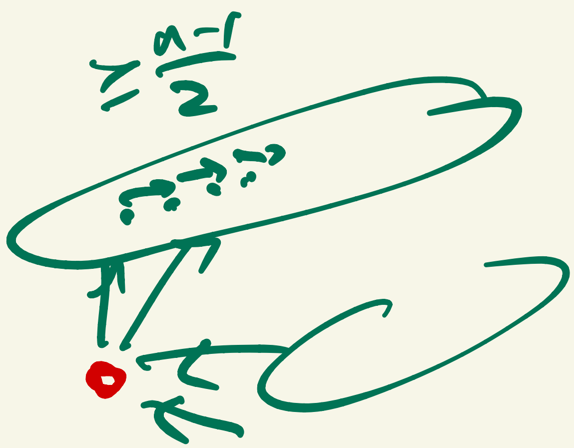
$$= \frac{n!}{|\mathrm{Aut}(\mathcal{H})|}$$

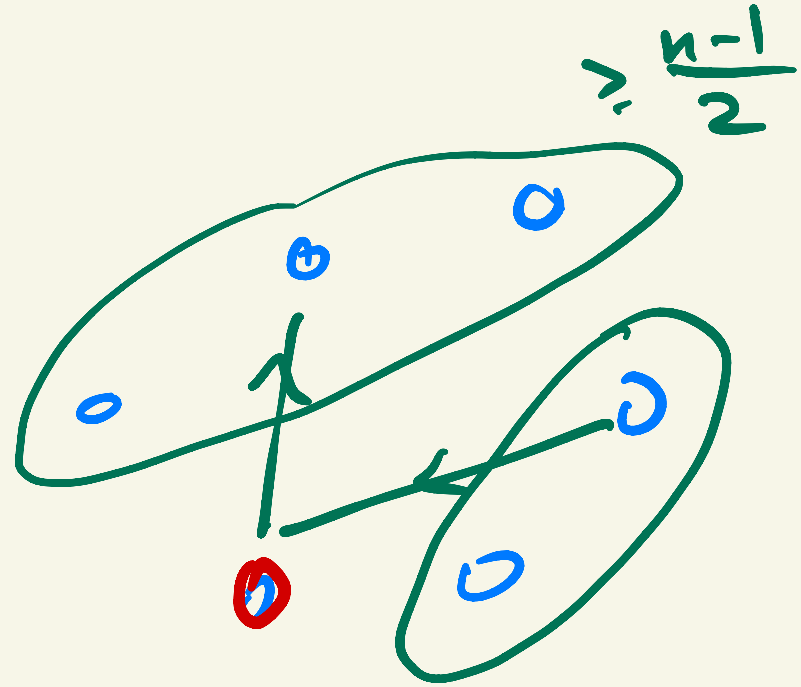
set of automorphism group

RUDI MATHON



$$(\forall a,b)\begin{cases} aRb \\ a=b \\ bRa \end{cases}$$





$f(n)$: $\min\limits_{R}$ max size of order subset

$$f(n) \geq 1 + f\left(\left\lceil \frac{n-1}{2} \right\rceil\right)$$

$$f(n) \geq \lceil \log_2(n+1) \rceil$$

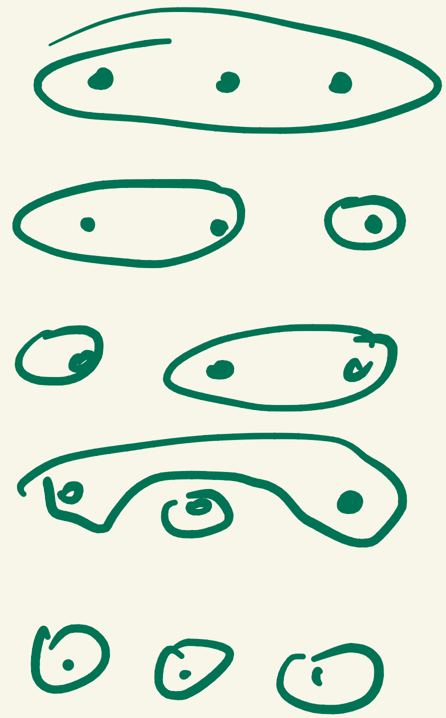
$$g(n) \geq 1 + g\left(\left\lceil \frac{n}{2} \right\rceil\right)$$

$$\Longrightarrow \quad g(n) \geq \lceil \log_2 n \rceil$$

B_n n^{th} Bell number:

number of partitions of $[n]$

$B_3 = 5$



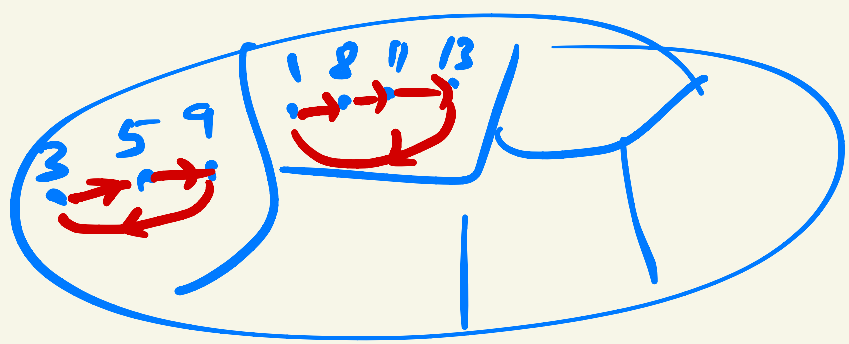
$\boxed{B_n \le n!}$

$\forall k \in [n]$

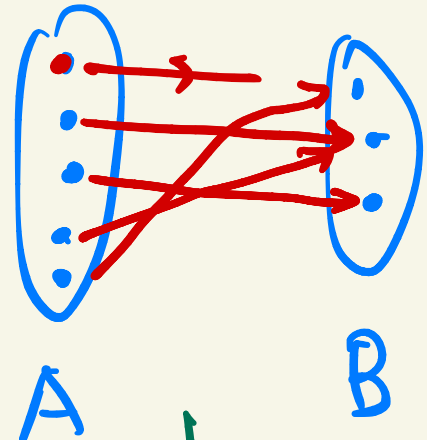
$\boxed{B_n \ge k^{n-k}}$

$\ln B_n \sim n \ln n$

inj: Partitions $\longrightarrow$ Perm's



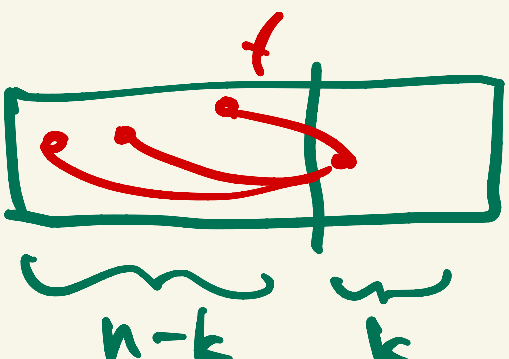
$$\Big|\underbrace{\{f : A \to B\}}_{B^A}\Big| = |B|^{|A|}$$



$$|B^A| = |B|^{|A|}$$

$$k^{n-k} = \Big|[k]^{[n-k]}\Big| \overset{?}{\leq} B_n \quad \checkmark$$

$$\underbrace{\bigsqcup \{i\} \cup f^{-1}(i)}_{\text{partition of } [n]}$$



$i \in [k]$ take $f^{-1}(i)$

$$[n-k] = \bigsqcup_{i \in [k]} f^{-1}(i)$$

inj: $[k]^{[n-k]} \longrightarrow$ partitions

$\forall k \in [n]$

$$k^{n-k} \leq B_n \leq n! \leq n^n$$

$$\therefore \quad (n-k)\ln k \leq \ln B_n \leq n \cdot \ln n$$

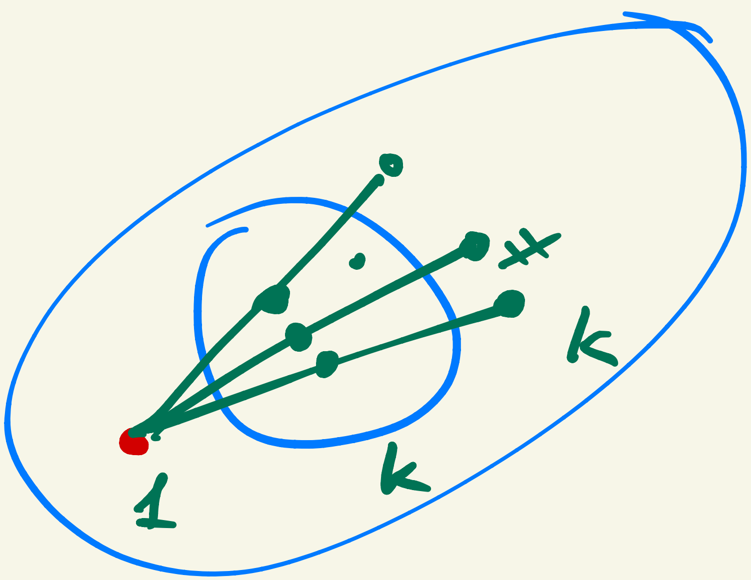
pick $k := \dfrac{n}{\ln n}$ $\qquad \dfrac{k}{n} \to 0$

$n-k \sim n$

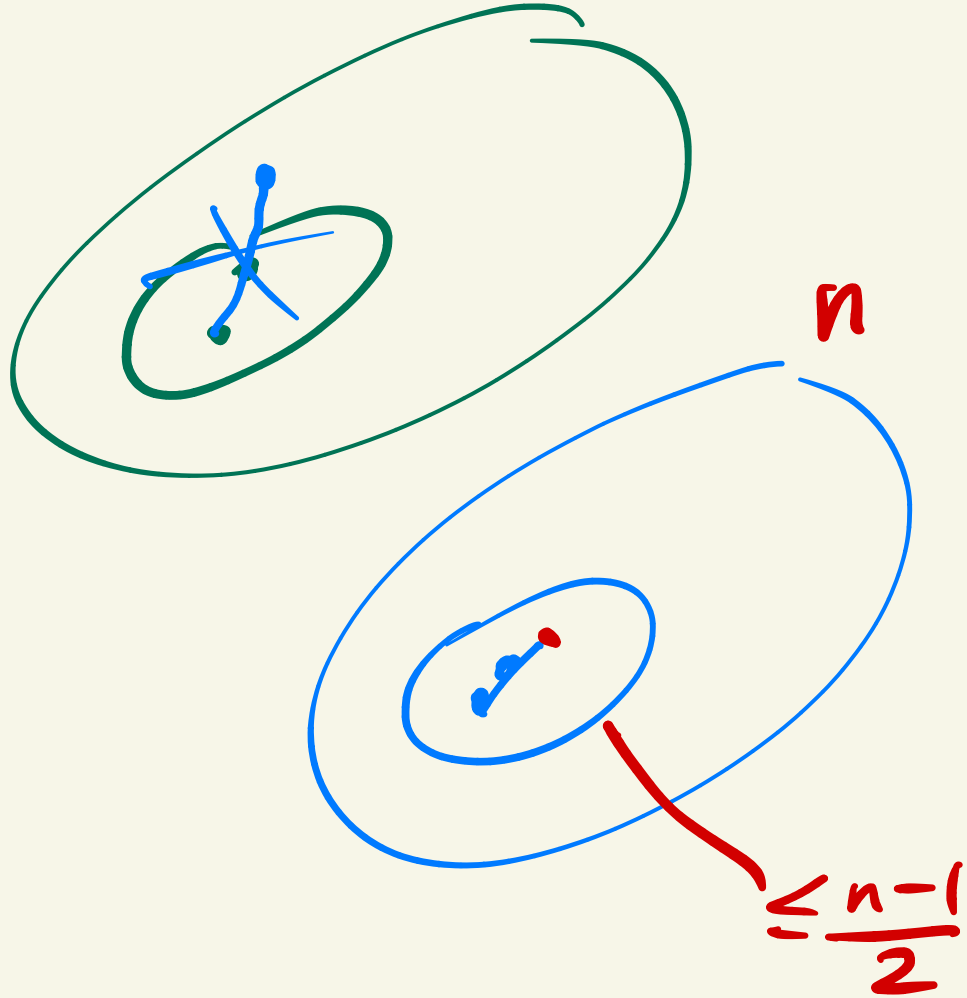
$$\ln k = \ln n - \underline{\ln\ln n} \sim \ln n$$

b/c $\dfrac{\ln n - \ln\ln n}{\ln n} = 1 - \underbrace{\dfrac{\ln\ln n}{\ln n}}_{\downarrow\, 0} \longrightarrow 1$

sub STS



$1 + k + k \leq n$ ✓



1.74

$a_n \sim b_n$

$a_n^n \not\sim b_n^n$

$a_n := e^{1/n}$

$b_n := 1$

$a_n^n = e$

$b_n^n = 1$

other sol:

$a_n = 1 + \frac{1}{n}$

$b_n = 1$

2.23 $\quad M$ incid. matrix of $\mathcal{G}$

diag. of $\quad M\cdot M^{tr} = [v_i\, v_j^T] \qquad v_i\cdot v_i^{tr} = \operatorname{rk} E_i$

$M^{tr}\cdot M$

$$M = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} = [w_1 \cdots w_n] \qquad w_j^{tr}\cdot w_j = \deg x_j$$

j^{th} vertex

$$\sum \operatorname{rk} E_i = Tr(M\cdot M^{tr}) = Tr(M^{tr}\cdot M) = \sum \deg x_j$$

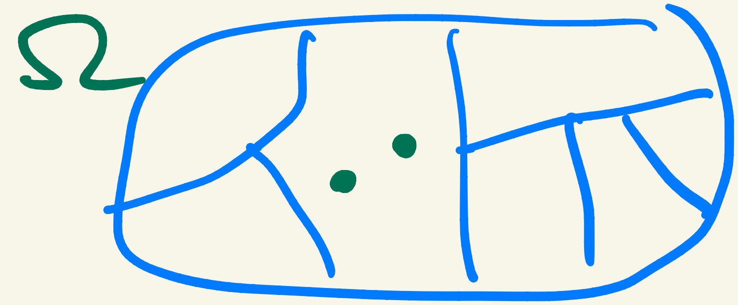
Handshake Thm

FUND THM of EQ RELATIONS

$\forall$ eq. rel. R

$\exists!$ partition Π

s.t. $R = \sim_\Pi$



Π partition

$\longrightarrow \sim_\Pi$ eq. rel.

$a \in \Omega \qquad [a] = \{ b \mid aRb \}$

Lemma If $[a] \cap [b] \neq \emptyset$ then $[a] = [b]$

Pf. $x \in [a] \iff a \in [x]$ symm. $\qquad a \in [a]$ refl

$x \in [a] \implies [x] \subseteq [a]$ but then $[a] \subseteq [x]$ b/c $a \in [x]$

$$\begin{tikzcd} a \arrow[r, "R"] \arrow[rr, bend right] & x \arrow[r, "R"] & y \end{tikzcd}$$

$\therefore\ x \in [a] \implies [a] = [x]$

$$\left.\begin{array}{l} c \in [a] \to [c] = [a] \\ c \in [b] \to [c] = [b] \end{array}\right\} \implies [a] = [b] \ \checkmark$$

$$a_{r+s} \geq a_r \cdot a_s$$

NTS $\exists \lim a_n^{1/n} = \sup a_n^{1/n}$

$$a_n = 5^{1/n}$$

$$a_{r+s} \qquad a_r \cdot a_s$$

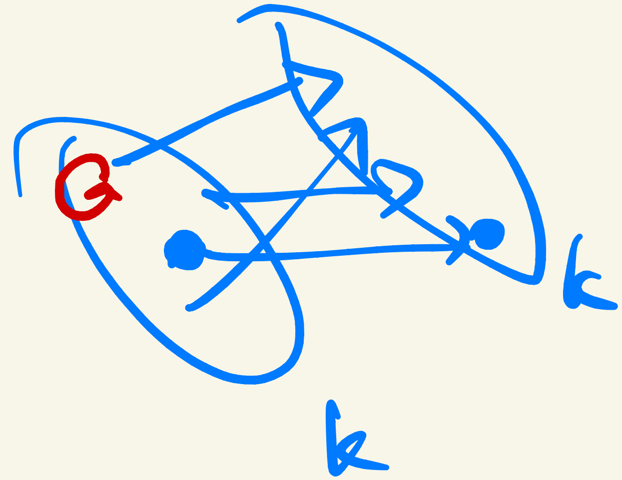
$$5^{\frac{1}{r+s}} \overset{?}{\gtreqless} 5^{\frac{1}{r}} \cdot 5^{\frac{1}{s}} = 5^{\frac{1}{r}+\frac{1}{s}} > 5^{\frac{1}{r+s}}$$

$$\frac{1}{2}+\frac{1}{2} > \frac{1}{5}$$

2.123

$$\#\text{trans rel} > 2^{\frac{n^2}{4}}$$

Pf If $n=2k$



$aRbRc$

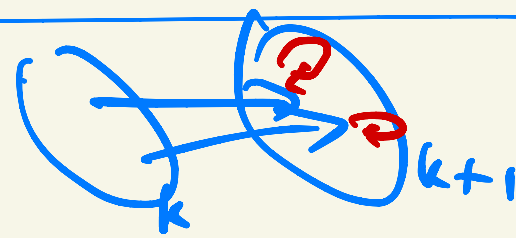
$\Downarrow$

$a=b$

$$\#\text{trans rel's} \geq 2^{k^2+k}$$

$$2^{k^2} = 2^{\frac{n^2}{4}}$$

$n=2k+1$



$$\# \geq 2^{k(k+1)+(k+1)} = 2^{(k+1)^2} > 2^{\frac{n^2}{4}}$$

dot product

$a, b \in \mathbb{F}^n$ $\qquad a \cdot b = \sum a_i b_i$

DEF $a \perp b$ if $a \cdot b = 0$

$(1, 2) \cdot (-2, 1) = 0$

$S \subseteq \mathbb{F}^n$

DEF S is totally isotropic if $(\forall a, b \in S)(a \perp b)$

incl. $a \perp a$

For some fields . . . $U \leq \mathbb{F}^n$

U tot isotr and $\dim U = \lfloor \frac{n}{2} \rfloor$