HONORS COMBINATORICS

2024-04-09

Bell numbers B_n : # partitions of $[n]$

partition $\pi = \{A_1 \dots A_k\}$

$$\begin{cases} A_i \neq \emptyset \\ \bigcup A_i = \Omega \\ i \neq j \Rightarrow A_i \cap A_j = \emptyset \end{cases}$$

$$\ln B_n \sim n \ln n$$

Generating function of a sequence $a_0, a_1, \ldots$

$$g(x) = \sum_{k=0}^{\infty} a_k x^k$$

Fib $F_0, F_1, \ldots$

EX.

$$Fib(x) = \sum_{k=0}^{\infty} F_k x^k = \frac{x}{1-x-x^2}$$

Exponential generating function

$$f(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k$$

$$\boxed{B(x) = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k = e^{e^x - 1}}$$

requires recurrence $B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k$

$$\Longrightarrow \quad B'(x) = e^x B(x)$$ ← differential equation

Dobiński's fmla

1877

THM

$$B_n = \frac{1}{e}\sum_{k=0}^{\infty} \frac{k^n}{k!}$$

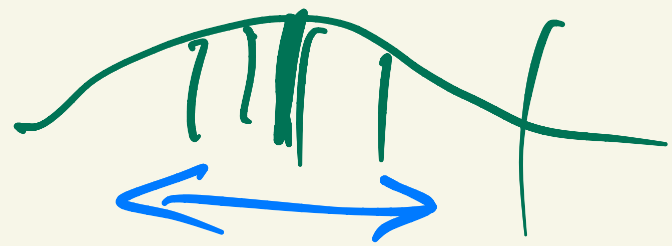
Proof: from $B(x) = e^{e^x - 1}$

$\lambda(n) = x$ where $x \ln x = n$

$\therefore \ \lambda(n) \sim \frac{n}{\ln n}$

$k(n) := \operatorname{argmax}_k \frac{k^n}{k!}$

$|k(n) - \lambda(n)| < 1$



$$\Longrightarrow \quad B_n \approx \frac{\lambda(n)^n}{\lambda(n)!} \approx \frac{\lambda(n)^n}{\left(\frac{\lambda(n)}{e}\right)^{\lambda(n)}} = \lambda(n)^{n-\lambda(n)} e^{\lambda(n)} =$$

close: $\lambda(n)^n \cdot e^{\lambda(n)-n}$

ASY, DMmini

Asymp. notation

$o, \omega, O, \Omega, \Theta$

$a_n = \Theta(b_n)$ if

$(\exists c, C > 0)(\forall$ all suff large $n)$

$c|b_n| \leq |a_n| \leq C|b_n|$

FINITE PROB SPACES

$(\Omega, \Pr)$

Ω finite, nonempty set

$\Pr : \Omega \to \mathbb{R}$ prob. distribution:

(i) $(\forall a \in \Omega)(\Pr(a) \geq 0)$

(ii) $\displaystyle\sum_{a \in \Omega} \Pr(a) = 1$

Ω : "sample space"

$\Pr$: probability

elements of Ω : elementary events

outcomes

$A \subseteq \Omega$: event

$$\Pr(A) = \sum_{a \in A} \Pr(a)$$

Ω	$\Pr$	X
1	1/2	10
2	1/3	1
3	1/6	-3

$\Omega = \{1, 2, 3\}$

$E(X) = \frac{1}{2}\cdot 10 + \frac{1}{3}\cdot 1 + \frac{1}{6}\cdot(-3)$

RANDOM VARIABLE

$X : \Omega \to \mathbb{R}$

If $\Pr$ uniform

$n = |\Omega|$

$(\forall a)\left(\Pr(a) = \frac{1}{n}\right)$

$E(X) = \frac{\sum X(a)}{n}$

simple average

EXPECTED (MEAN) VALUE

DEF

$$E(X) = \sum_{a\in\Omega} X(a)\cdot \Pr(a)$$

weighted average

$\min X \le E(X) \le \max X$

Ex. $X, Y : \Omega \to \mathbb{R}$ on same prob. space

$$E(X+Y) = E(X) + E(Y)$$
$$E(cX) = cE(X)$$

linearity of expectation

$$\Leftrightarrow \quad Z = \sum_{i=1}^{k} c_i X_i \qquad c_i \in \mathbb{R}$$

$$\Rightarrow E(Z) = \sum c_i E(X_i)$$

linearity of expectation

A, B sets

set of functions

$$B^A = \{ f : A \to B \} \qquad |B^A| = |B|^{|A|}$$

domain codomain

Random variables $\in \mathbb{R}^\Omega$ $\leftarrow$ vector space

$$\dim(\mathbb{R}^\Omega) = |\Omega| =: n$$

b/c basis: $(\delta_a \mid a \in \Omega)$

$$\delta_a(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x \neq a \end{cases}$$

$x \in \Omega$

$\bullet$ ___ ___

$\Omega = \mathbb{B}^n$ $\qquad \mathbb{B} = \{0, 1\}$

n coin tosses

X : # heads $\qquad |\text{Range}(X)| = n+1$

thm

EX

$$E(X) = \sum_{y \in \text{Range}(X)} y \cdot P(X = y)$$

prob. distrib. on Range(X)

"$X = y$" means $\{a \in \Omega \mid X(a) = y\} \subseteq \Omega$

$= X^{-1}(y)$

DEF Indicator variable: Range $\subseteq \mathbb{B} = \{0,1\}$

COR If X is an indicator variable then

$$E(X) = 1 \cdot \Pr(X=1) + 0 \cdot \Pr(X=0) = \Pr(X=1)$$

1-to-1 corr. betw. events and indic. var's

let $A \subseteq \Omega$ event

$$\vartheta_A : \Omega \to \mathbb{R} \qquad \vartheta_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in \Omega \setminus A \end{cases}$$

$x \notin A$

$x \in \overline{A}$

\vartheta

$$E(\vartheta_A) = \Pr(A)$$

Bernoulli trial with prob $\underline{p}$ of success:

indicator var $\quad P(X=1)=p$

$\qquad\qquad\qquad P(X=0)=1-p$

flipping a biased coin with $P(\text{heads})=p$

sequence of n Bernoulli trials
w prob $\underline{p}$ of success

$X :=$ # successes

$$\boxed{E(X) = n \cdot p}$$

if coin flips indep.

$\Omega = \mathbb{B}^n$ $\quad$ $\mathbb{B} = \{0,1\}$

$x \in \mathbb{B}^n$ $\quad$ x (0,1) string

X: random string

$h(x)$: # heads $= \#\, 1$'s

$= \sum x_i \qquad x = \overline{x_1 \ldots x_n}$

$X = \overline{X_1 \ldots X_n}$

$P(X = \underline{x})$

given string

0111001

$$P(X_i = x_i) = \begin{cases} p & \text{if } x_i = 1 \\ 1-p & \text{if } x_i = 0 \end{cases}$$

$$P(X = \underline{x}) = p^{h(x)} \cdot (1-p)^{n-h(x)}$$

$$Z := \#\text{ heads in } X = h(X)$$

$$E(Z) = \sum_{y=0}^{n} y \cdot \underbrace{P(Z=y)}$$

$$\sum_{y=\cancel{0}\,1}^{n} y \cdot \binom{n}{y} \cdot p^{y} \cdot (1-p)^{n-y} =$$

$$y \geq 1$$

$$= n \sum_{y=1}^{n} \binom{n-1}{y-1} \cdot p^{y} (1-p)^{n-y}$$

$$y \cdot \binom{n}{y} = n \binom{n-1}{y}$$

$$\frac{n(n-1)\cdots(n-y+1)}{y\,(y-1)!}$$

$$t := y-1$$

$$= np \underbrace{\sum_{t=0}^{n-1} p^{t} (1-p)^{n-1-t} \cdot \binom{n-1}{t}}_{\text{Binomial Thm } (p+(1-p))^{n-1} = 1} = np$$

2^{nd} proof (MUCH better)

$\underline{Z}$ = #heads $= \sum X_i$ | X_i indicates "i^{th} coin: heads"

$$E(Z) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} \Pr(\underbrace{\quad\quad}_{p}) = \underline{np}$$

works w/o assumption of independence

X_i Bern. trial w prob p_i of succ, $Z = \sum X_i$

$$\Rightarrow E(Z) = \sum_{i=1}^{n} p_i$$

Latin squares

$n \times n$

$M = (m_{ij})$

$|\Sigma| = n$ symbols

$\forall$ row
$\forall$ col
each symbol
exactly once

$$\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$$

$n \times n$

$$\begin{pmatrix} a_0 & \cdots & a_{n-1} \\ a_{n-1} & a_0 \cdots & a_{n-2} \\ & \vdots & \end{pmatrix}$$

$L_1 = (m_{ij})$ $\quad L_2 = (q_{ij})$ two L. Sq's

<u>orthogonal</u> if

$$\left| \{ (m_{ij}, q_{ij}) \mid i,j \in [n] \} \right| = n^2$$

each pair exactly once

$$\begin{pmatrix} a & \boxed{b} & c \\ b & c & a \\ c & a & b \end{pmatrix} \perp \begin{pmatrix} a & \boxed{c} & b \\ b & a & c \\ c & b & a \end{pmatrix}$$

← marked location of pair (b,c)

EULER's 36 officers problem: $\exists$? pair of 6×6 orthogonal Latin squares