HONORS
COMBINATORICS

2024.04.11

1

Ω finite set $\quad |\Omega| = n$

$\mathcal{P}(\Omega) = \{ A \mid A \subseteq \Omega \}$

powerset

$|\mathcal{P}(\Omega)| = 2^n$

$A \subseteq B$

$A \subset B$ proper subset

$A \| B$ incomparable: $\begin{cases} A \not\subseteq B \\ B \not\subseteq A \end{cases}$

$\mathcal{P}(\Omega)$ $\quad |\Omega| = n$

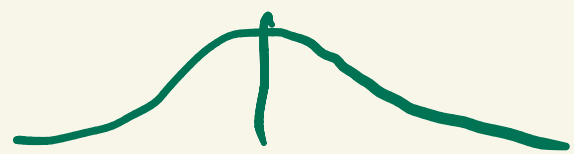
Chain $\quad A_1 \subset A_2 \subset \cdots \subset A_k$

$n+1 = \max k$

Antichain: $\quad B_1 \;\dots\; B_m$

s.t. $(\forall i \neq j)(B_i \parallel B_j)$

max size of antichain



Examples: uniform

k-uniform : $\le \binom{n}{k}$ $\quad \max = \binom{n}{\lfloor n/2 \rfloor}$

THEOREM (Sperner)

If $\mathcal{A} \subseteq \mathcal{P}(\Omega)$, $\mathcal{A}$ antichain
(Sperner family)

$$\Longrightarrow \quad |\mathcal{A}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

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linear order on Ω

$\uparrow$

$n!$

$\underbrace{|a_1|a_2|}_{\text{prefix}} \cdots |a_n|$

prefixes of a lin. order: $n+1$

Proof: Lubell's permutation method

$\mathcal{A} = \{A_1 \ldots A_m\}$

Let π be a random lin. order

$X = \#\{i \mid A_i \text{ is a prefix of } \pi\}$

r.v.

$X \leq 1$ always ← Sperner family

$\therefore \quad E(X) \leq 1$

$X = Y_1 + \cdots + Y_m$

$Y_i = \begin{cases} 1 & \text{if } A_i \text{ pref of } \pi \\ 0 & \text{o/w} \end{cases}$

indicator

$$1 \geq E(X) = \sum_{i=1}^{m} E(Y_i) = \sum_{i=1}^{m} P(A_i \text{ pref of } \pi) = \sum_{i=1}^{m} \frac{1}{\binom{n}{|A_i|}}$$

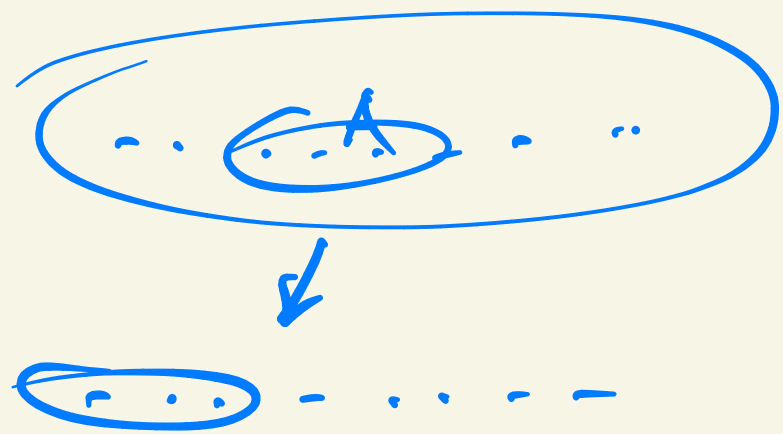
$[n]$

$A \subseteq \Omega$ $\qquad$ π random perm (linear order) $\qquad$ 5

$$\Pr(A \text{ is a prefix of } \pi) = \Pr(\pi(A) \text{ is } [k])$$

$|A| = k$

$$= \frac{1}{\binom{n}{k}} = \frac{1}{\binom{n}{|A|}}$$



LEMMA BLYM inequality

If $\{A_1, \dots, A_m\}$ is a Sperner family

then
$$\sum_{i=1}^{m} \frac{1}{\binom{n}{|A_i|}} \leq 1$$

COR
$$m \leq \binom{n}{\lfloor n/2 \rfloor}$$

Pf
$$1 \geq \sum_{i=1}^{m} \frac{1}{\binom{n}{|A_i|}} \geq \sum_{i=1}^{m} \frac{1}{\binom{n}{\lfloor n/2 \rfloor}} = \frac{m}{\binom{n}{\lfloor n/2 \rfloor}}$$

✓

CH Equality in Sperner's Thm

$\iff \mathcal{A}$ is uniform of rank $\lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$

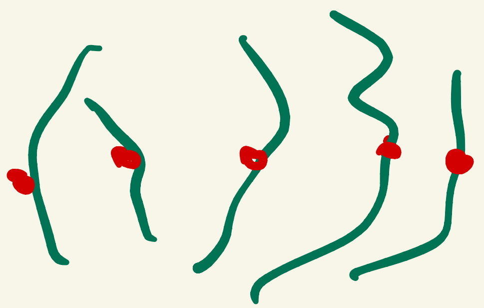
CH Equality in BLYM holds if and only if $\mathcal{A}$ is max. uniform

n

THM $\exists$ chains $C_1 \dots C_{\binom{n}{\lfloor n/2 \rfloor}}$

that is the min possible # of chains

such that $\mathcal{P}(\Omega) = \bigcup_i C_i$



If $\mathcal{A}$ is antichain and $\mathcal{C}$ is a chain cover

then $|\mathcal{A}| \underset{\text{obvious}}{\leq} |\mathcal{C}|$ $\therefore \max |\mathcal{A}| \leq \min |\mathcal{C}|$

Surprise: $=$

Finite proj. plane of <u>order</u> n :

(P, L, I) $\quad \forall p \ \deg(p) = n+1$

point lines indic. rel $\quad \forall \ell \ \mathrm{rk}(\ell) = n+1$

$$|L| = |P| = n^2+n+1$$

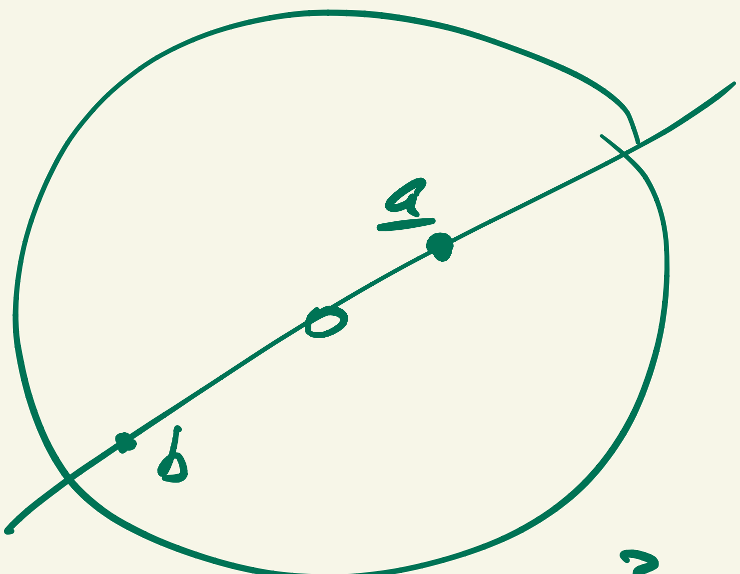
Example: Fano plane has order 2

For what n does a p.p. of order n exist?

<u>YES</u> if $n = \underbrace{p^k}_{q}$ prime power

<u>Galois planes</u> from Galois fields: $\mathbb{F}_q$

GF(q)



homogeneous coordinates

$$(a_1 : a_2 : a_3) = (b_1 : b_2 : b_3)$$

$$\neg(a_1 = a_2 = a_3 = 0)$$

$$\mathbb{R}^3 \setminus \{0\} \big/ \mathbb{R}^\times \qquad \mathbb{R}^\times = \mathbb{R} \setminus \{0\}$$

Field $\mathbb{F}$

$\longrightarrow \mathbb{P}\mathbb{F}^2 = \mathbb{F}^3 \setminus \{0\} \,/\, \mathbb{F}^\times$

projective plane over $\mathbb{F}$

$\mathbb{F}^\times = \mathbb{F} \setminus \{0\}$

$p = (a_1 : a_2 : a_3) \ni \underline{p}$

Lines:

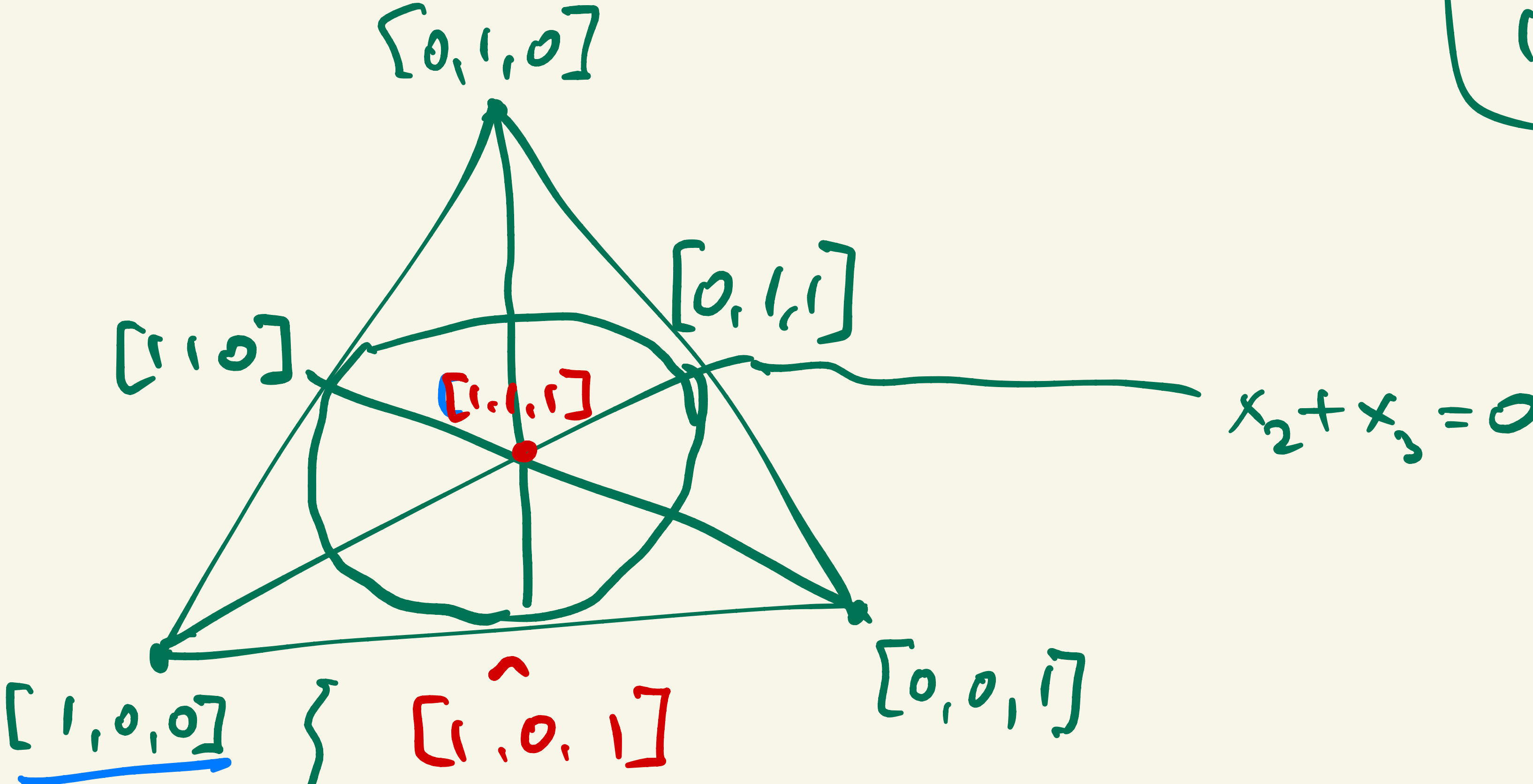
$\underline{c_1 x_1 + c_2 x_2 + c_3 x_3} = 0$

$\ell = (c_1 : c_2 : c_3) \ni \underline{\ell}$

$p \text{ —•— } \ell$

$\Updownarrow$

$\underline{p} \perp \underline{\ell}$



ℓ

$$\begin{array}{ccc} 1 & 0 & 0 \\ c_1x_1 + & c_2x_2 + & c_3x_3 = 0 \end{array}$$

$c_1=0$

$c_3=0$

ℓ: $x_2=0$

BRUCK – RYSER 1949

If $\exists$ p.p. of order n

and $n \equiv 1$ or $2 \bmod 4$

then $(\exists x, y)(n = x^2 + y^2)$

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6 ✗

10 ? NO ✓

12 ??

14 ✗

15 ???

$n \times n$ L. Squares, pwise $\perp$

$\Longrightarrow \quad \leq n-1$

and $n-1$ exist

$\Longleftrightarrow \exists$ p.p. order