COMBINATORICS 2024-04-12

PROBLEM SESSION

4.51

$U \le \mathbb{F}^n$

subspace

$$\dim U + \dim U^\perp = n$$

Pf basis of U: $b_1, \ldots, b_k$ row vectors

$$M = \begin{bmatrix} b_1 \\ \vdots \\ b_k \end{bmatrix} \cdot \underline{x} = \begin{bmatrix} b_1 \cdot x \\ \vdots \\ b_k \cdot x \end{bmatrix}$$

$\underline{x} \in U^\perp \iff x \perp b_1 \ldots b_k \iff \text{RHS} = 0$

$\iff x \in \mathrm{Null}(M)$

i.e. $U^\perp = \mathrm{Null}(M)$ $\therefore \dim U^\perp + \underbrace{\mathrm{rk}\, M}_{k} = n$

U is totally isotr if $U \perp U$

i.e. $U \leq U^{\perp}$

dim $k \leq n-k$ $\therefore$ $2k \leq n$ $\therefore$ $k \leq \lfloor \frac{n}{2} \rfloor$

4.59 If $\mathbb{F} = \mathbb{F}_2, \mathbb{F}_5, \mathbb{C}$

then $\forall n$ $\exists$ totally isotr subsp of dim $\lfloor \frac{n}{2} \rfloor$

$(x,y) \in \mathbb{F}^2$ isotropic if $x^2+y^2=0$, $(x,y) \neq \underline{0}$

$\Longleftrightarrow (\exists x)(x^2+1=0)$ true for these 3 fields

$i^2 = -1$

$$\begin{pmatrix} 1 & & \\ i & & \\ & 1 & \\ & i & \\ & & 1 \\ & & i \end{pmatrix}$$

columns span k-dim tot. isotr.

2.49 $A_1, \ldots, A_n \subseteq [n]$ $\quad \underline{m = n}$

all distinct

then $(\exists i)(A_1 \setminus \{i\}, \ldots, A_n \setminus \{i\})$ also all distinct

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proof 1 induction

if removal of $\{n\}$ keeps them all distinct ✓

o/w $B_j = A_j \setminus \{n\}$ are not all distinct.

Let $C_1 \ldots C_k$ be as many distinct B_j as there are

the rest: $D_1 = C_1 \ldots D_\ell = C_\ell$ $\quad \ell \le k$

Apply IH to $C_1 \ldots C_k \subseteq [n-1]$ $\quad k \le n-1$

$\therefore (\exists i)(\text{the } C_j \setminus \{i\} \text{ all distinct})$

Pf #2 $v_1 \dots v_n \in \mathbb{F}^n$ incid. vectors

$v_j^{(i)}$: v_j with i^{th} coord. removed

$$\begin{array}{c} i \\ 1101\cancel{1}0001 \end{array}$$

what does it mean

if $v_j^{(i)} = v_k^{(i)}$ then $v_j - v_k = \pm e_i$
$\underbrace{j \neq k}$

$$\begin{array}{c} i \\ (000100) \end{array}$$

Pf by contradiction: suppose none of i work

then $(\forall i)(\exists j \neq k)(e_i = v_j - v_k)$

$\therefore \text{span}(v_j - v_k \mid j,k \in [n]) = \mathbb{F}^n$

but all $v_j - v_k \in$

$\text{Span}(v_1 - v_2, v_1 - v_3, \dots, v_1 - v_n)$

2.158 $\alpha(\mathcal{H})$

$\mathcal{H} = (V, \mathcal{E})$ $\mathcal{E} \neq \emptyset$

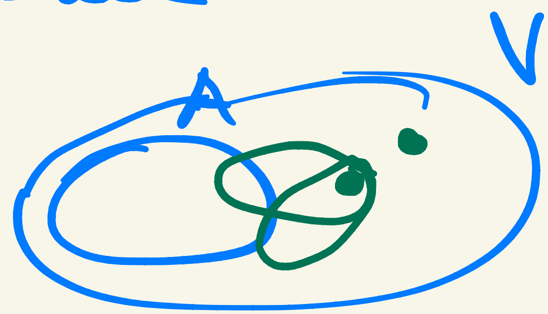
r-uniform, regular

$|V| = n$ $|\mathcal{E}| = m$

Claim: $\boxed{\alpha \leq n\left(1 - \frac{1}{r}\right)}$

degree $= d > 0$

$$\underline{rm = nd} \leftarrow \text{Handshake}$$



$A \subseteq V$ indep

$$d \cdot |\bar{A}| \geq m = \frac{nd}{r}$$

$$d \cdot (n - \alpha) \geq \frac{nd}{r}$$

$d \neq 0 \Rightarrow$ $n - \alpha \geq \frac{n}{r}$ ✓

Eventown thm:

If $A_1 \cdots A_m \subseteq [n]$

distinct and $(\forall i,j)(|A_i \cap A_j| = \text{even})$

$\Longrightarrow m \leq 2^{\lfloor \frac{n}{2} \rfloor}$

Incid. vectors $v_1 \cdots v_m$ over $\mathbb{F}_2$

$(\forall i,j)(v_i \cdot v_j = 0)$

If our system is maximal then it is a subspace

b/c $S \perp S \Longrightarrow \operatorname{Span} S \perp \operatorname{Span} S$

$S \leq \mathbb{F}^n$ $\quad S \perp S$: S totally isotropic

$\therefore \dim S \leq \lfloor \frac{n}{2} \rfloor \quad \therefore |S| \leq 2^{\lfloor \frac{n}{2} \rfloor}$

5.65

Fekete's Lemma

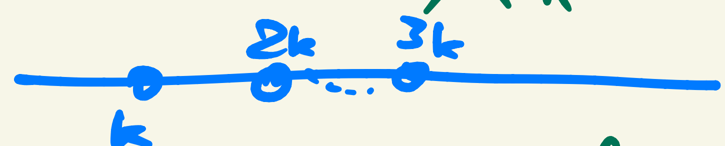
supermult. seq. of positive reals:

$a_1 \quad a_2 \quad \dots \qquad a_{k+\ell} \ge a_k \cdot a_\ell$

then $\lim a_k^{1/k}$ exists $= \sup_k a_k^{1/k}$

NTS $\forall T < \sup a_k^{1/k} \quad (\exists n_0)(\forall n > n_0)(a_n^{1/n} \ge T)$

by def sup $(\exists k)(a_k^{1/k} > T)$



$a_{k\ell}^{\frac{1}{k\ell}} \ge a_k^{1/k} \qquad a_{k\ell} \ge a_k^{\ell} > T$

$n = k\ell - r$

$0 \le r < k$

$\cancel{a_n \ge a_k^{\ell} \cdot a_1^{r}}$

$a_n \cdot a_1^{r} \ge a_k^{\ell}$

$a_n^{1/n} \cdot \underbrace{a_1^{r/n}} \ge a_k^{\ell/n} \ge \underbrace{a_k^{1/k} > T}$

5.35 Affine lines over $\mathbb{F}_3$

$$= \{ \{\underline{a}, \underline{b}, \underline{c}\} \mid \underline{a}, \underline{b}, \underline{c} \in \mathbb{F}_3^n,\ \underbrace{\underline{a} \neq \underline{b},\ \underline{a} + \underline{b} + \underline{c} = \underline{0}} \}$$

$\Leftrightarrow$ $\underline{a}, \underline{b}, \underline{c}$ distinct
$\underline{a} + \underline{b} + \underline{c} = \underline{0}$

① ℓ aff. line $\Longrightarrow$

$$\ell = \underline{u} + \{0, \underline{v}, 2\underline{v}\} \qquad \underline{v} \neq \underline{0}$$

$$= \{\underline{u}, \underline{u} + \underline{v}, \underline{u} + 2\underline{v}\}$$

adding them up: $3\underline{u} + 3\underline{v} = \underline{0} + \underline{0} = \underline{0}$

② $\{a, b, c\}$ as above. Claim $\{0, b-a, c-a\} \leq \mathbb{F}_3^n$ 1-dim

i.e. $c - a = 2(b-a) \Leftarrow$

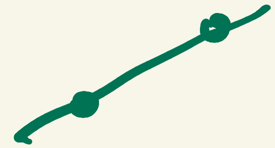
$c - a = 2b - 2a \Leftarrow$

$a + b + c = a - 2b + c = 0$

5.29

affine lines in $\mathbb{F}_q^n$

lemma :



$$\therefore \quad \frac{\binom{q^n}{2}}{\binom{q}{2}} = \frac{q^n(q^n-1)}{q(q-1)} = q^{n-1}\cdot\frac{q^n-1}{q-1}$$