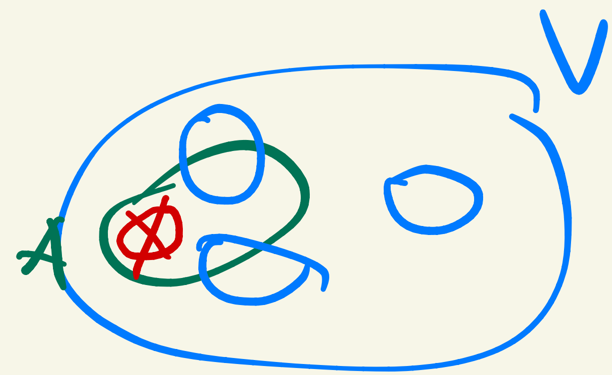
$\mathcal{H} = (V, \mathcal{E})$ hypergraph

DEF <u>independent</u> set $A \subseteq V$ s.t.

$$(\forall E \in \mathcal{E})(E \not\subseteq A)$$



DEF <u>Legal coloring</u> of $\mathcal{H}$

$f : V \to \Sigma$ ← set of "colors"

s.t. no edge is <u>monochromatic</u>, i.e.,

$$(\forall E \in \mathcal{E})(|f(E)| \geq 2)$$

DEF **Legal coloring of $\mathcal{H}$**

$f : V \longrightarrow \Sigma$ $\Leftarrow$ set of "colors"

s.t. no edge is monochromatic, i.e.,

$$(\forall E \in \mathcal{E})(|f(E)| \geq 2)$$

Chromatic number $\chi(\mathcal{H}) =$ min # colors needed for a legal coloring

\chi

$\alpha(\mathcal{H}) =$ max size of indep. set.

\alpha

$\boxed{\text{HW}}$ $\alpha(\mathcal{H}) \cdot \chi(\mathcal{H}) \geq n$ $n = |V|$

If $(\exists E)(|E| \le 1)$ then $\chi(\mathcal{H})$ undefined

Complete r-uniform hypergraph = clique

on n vertices: $K_n^{(r)} = \left([n], \binom{[n]}{r}\right)$

$|V| = n$ $\qquad K_V^{(r)} = \left(V, \binom{V}{r}\right)$

$r \ge 2$

$$\chi\left(K_n^{(r)}\right) = \left\lceil \frac{n}{r-1} \right\rceil$$

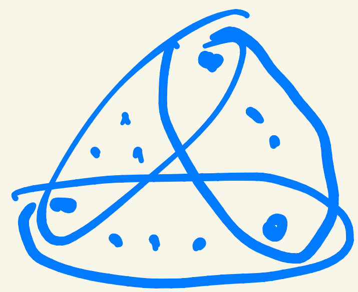
for fixed r $\qquad \Omega(n)$ $\qquad$ i.e. $(\exists c > 0)(\forall$ suffic. large $n)(\chi \ge cn)$

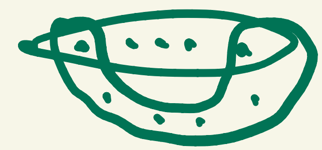
Can χ be large if $\forall E \neq F \in \mathcal{E}$

$|E \cap F| \le 1$? (e.g. STS)

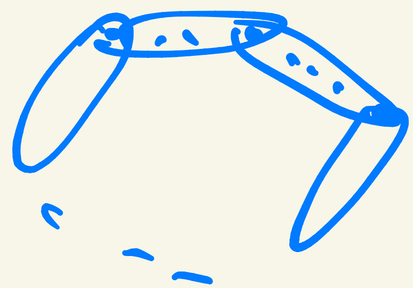
---

triangle in $\mathcal{H}$:





cycle of length k



Can χ be large if

no cycle of length $\le k$

PAUL ERDŐS

$(\forall r \geq 2, k, s)(\exists$ r-unif. hyp. without cycles of length $\leq k$ s.t. $\chi \geq s)$

1957 graph
~1960 hyp.

PROBABILISTIC METHOD

proof of existence w/o explicit constr.

ERDÖS

$\boxed{\text{HW}}$ If $\mathcal{H}$ is r-uniform and

$m \leq 2^{r-1}$ then $\chi \leq 2$

#edges

2-colorable

Pf: take random coloring $|\Omega| = 2^n$

sample space

show: $P(\text{random coloring is legal}) > 0$

$m(r)$: min # ~~vertices~~ edges of r-unif hyp. $\chi \geq 3$

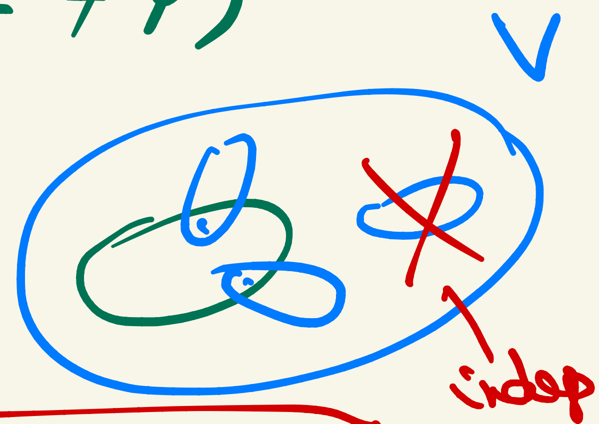
$m(r) > 2^{r-1}$

ERDÖS: $m(r) = O(r^2 \cdot 2^r)$

big-Oh

covering / hitting set } of $\mathcal{H}$: $C \subseteq V$

s.t. $(\forall E \in \mathcal{E})(C \cap E \neq \emptyset)$



Covering number

$\tau(\mathcal{H}) = \min |C|$

$$\boxed{\alpha + \tau = n}$$

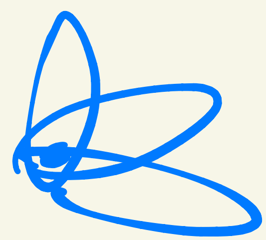
\tau

α is NP-hard to approximate within a factor of $n^{1-\varepsilon}$

$> n^{1-\varepsilon}$ $\quad$ $\leq n^{\varepsilon}$

LOVÁSZ 1978

greedy cover algorithm approximates τ within a factor of $1+\ln(\deg_{max})$

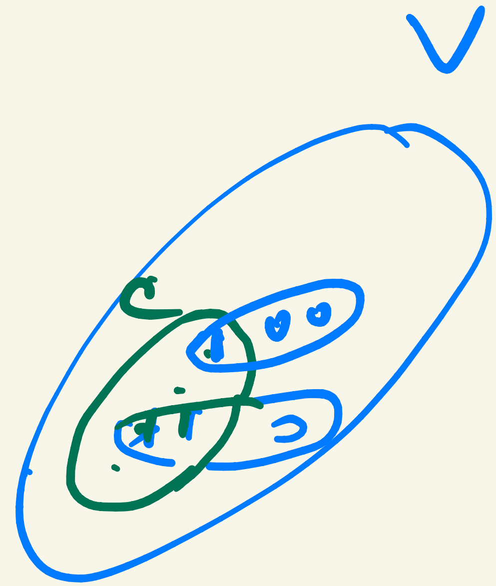


Fractional cover

$$c: V \to \mathbb{R}$$

s.t. $(\forall x \in V)(c(x) \geq 0)$

$(\forall E \in \mathcal{E})\left(\sum_{x \in E} c(x) \geq 1\right)$



$$\tau^* = \min_{c} \sum_{x \in V} c(x)$$

c: fractional cover

$$\boxed{\tau^* \leq \tau}$$

LOVÁSZ: ① $\tau \leq \tau^* \cdot (1 + \ln \deg_{max})$

② $\text{greedy-}\tau \leq$ — " —

② ⇒ ①

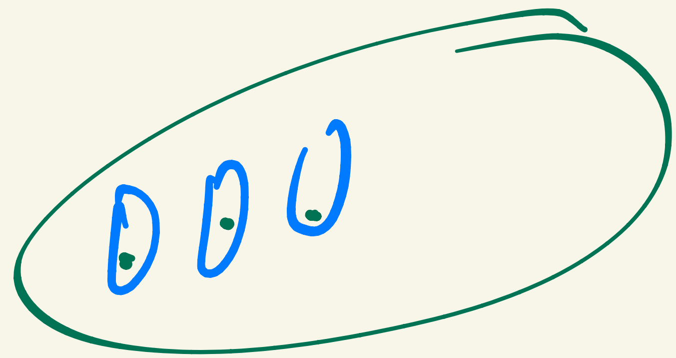
Matching in a hypergraph is a set of disjoint edges

$$\mathcal{M} \subseteq \mathcal{E} \qquad (\forall E \neq F \in \mathcal{M})(E \cap F = \emptyset)$$

---

matching number

$\nu(\mathcal{H})$ = max size of match.



Thm

$\boxed{\text{Do}}$ $$\nu \leq \tau$$

$$\tau\left(\mathcal{K}_n^{(r)}\right) = n-r+1 \qquad r \geq 2$$

$$\nu\left(\mathcal{K}_n^{(r)}\right) = \left\lfloor \frac{n}{r} \right\rfloor$$

$r=2$: poly time

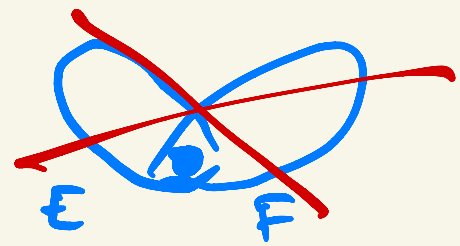
$r \geq 3$ NP-hard

Fractional matching:

$$g : \mathcal{E} \to \mathbb{R}$$

$$(\forall E \in \mathcal{E})(g(E) \geq 0)$$

$$(\forall v \in V)\left(\sum_{\substack{E \\ v \in E}} g(E) \leq 1\right)$$



$$g(E) + g(F) \leq 1$$

$$\nu^* = \max_g \sum_{E \in \mathcal{E}} g(E)$$

g fract. matching

\nu

$$\nu^* \geq \nu$$

$\boxed{\text{HW}}$ $\nu^* \leq \tau^*$

$\nu \leq \tau$ ✓

$$\nu \leq \nu^* \leq \tau^* \leq \tau$$

$$\nu^* = \tau^*$$

Special case of the DUALITY THEOREM of LINEAR PROGRAMMING

Linear programming LP

linear constraints: $a_{11}x_1 + \cdots + a_{1n}x_n \le b_1$

$$\vdots$$

$$a_{m1}x_1 + \cdots + a_{mn}x_n \le b_n$$

linear objective fctn

$$c_1 x_1 + \cdots + c_n x_n$$

$$\begin{matrix}\max\\ \min\end{matrix}\{\text{obj. fctn} \mid \text{constraints}\}$$

STANDARD FORM: $x_i \ge 0$

<u>DEF</u>

$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad \underline{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$\underline{x} \geq \underline{y} \text{ if } (\forall i)(x_i \geq y_i)$$

partial order

$$\underbrace{A}_{m \times n \text{ matrix}} \underbrace{\underline{x}}_{\text{unknown vector in } \mathbb{R}^n} \leq \underbrace{\underline{b}}_{\in \mathbb{R}^m}, \quad \underline{x} \geq \underline{0}$$

obj. fctn. $\underline{c} \cdot \underline{x}$

STANDARD LP

$$\max \{ \underline{c} \cdot \underline{x} \mid \underline{x} \geq \underline{0}, \ A\underline{x} \leq \underline{b} \}$$