HONORS COMBINATORICS | 2024-04-18

DEF **Permutation** of a set A | acts on A

$\pi : A \to A$ bijection

set of permutations acting on A:

$\mathrm{Sym}(A)$ group — assoc, identity 1_A, inverse

$x^{1_A} = x$

A = permutation domain

$A \xrightarrow{\sigma} A \xrightarrow{\tau} A$

$x \longmapsto x^\sigma \longmapsto (x^\sigma)^\tau = x^{(\sigma\tau)}$

$(x^\sigma)^{\sigma^{-1}} = x$

$n = |A|$

order: $|\mathrm{Sym}(A)| = n!$

degree: $|A| = n$

$\sigma\sigma^{-1} = \sigma^{-1}\sigma = 1_A$

order $|G|$

degree $|A|$

G is a permutation group acting on A

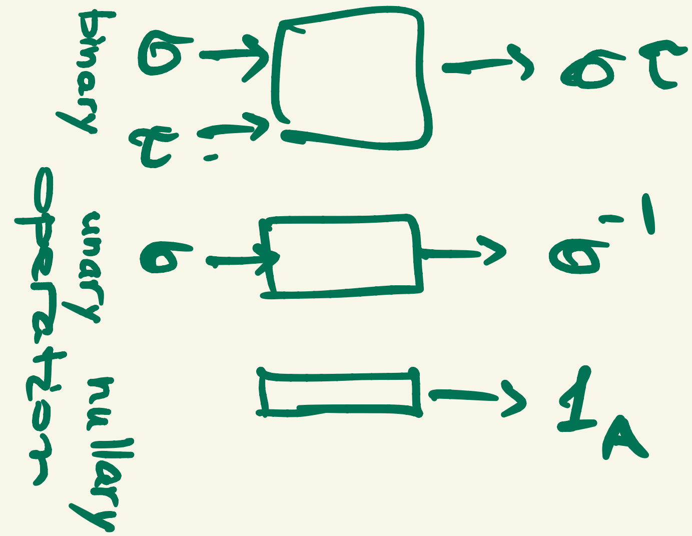
if $G \leq \mathrm{Sym}(A)$

subgroup

1. $G \subseteq \mathrm{Sym}(A)$
2. G is closed under multiplication

   $\sigma, \tau \in G \Longrightarrow \sigma\tau \in G$
3. $1_A \in G$ ← closed under identity
4. $(\forall \sigma \in \mathrm{Sym}(A))(\sigma \in G \Longrightarrow \sigma^{-1} \in G)$

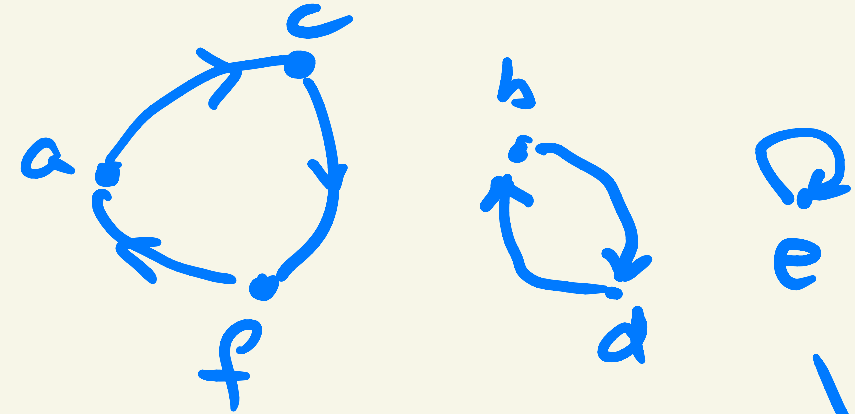
   closed under inverses



If $A = [n]$ $\quad \mathrm{Sym}([n]) = S_n$

x	x^σ
a	c
b	d
c	f
d	b
e	e
f	a

cycle decomposition



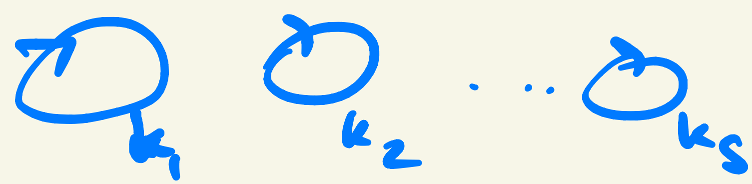
fixed point

order of σ is k if $\sigma^k = 1_A$
and for $0 < l < k$ $\quad \sigma^l \neq 1_A$

$\mathrm{ord}(\sigma)$

$\mathrm{ord}(1_A) = 1$

σ:

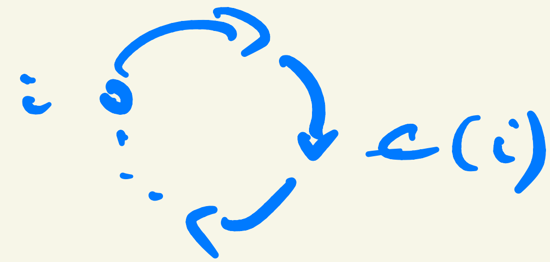


$\mathrm{ord}(\sigma) = \mathrm{l.c.m}(k_1, \dots, k_s)$

for $i \in A$ $\quad c(i) :=$ length of the cycle of i

$1 \le k \le n$

$$\Pr_{\sigma \in S_n}\left(c(1) = k\right) = P_k$$



$k = 1$ $\quad P_1 = \dfrac{(n-1)!}{n!} = \dfrac{1}{n}$

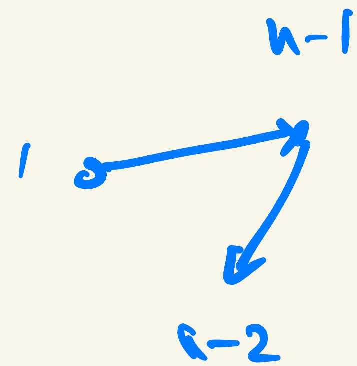
$k = n$ $\quad P_n = \dfrac{(n-1)!}{n!} = \dfrac{1}{n}$

$\dfrac{1}{n!}$

$\dfrac{1}{(n-1)!}$

$\boxed{\text{HW}}$ $(\forall k \in [n])\left(P_k = \frac{1}{n}\right)$

$\boxed{\text{HW}}$ $E(\#\text{cycles}) \sim \ln n$



$$\mathcal{H} = (V, \mathcal{E})$$

Automorphism: self-isomorphism

$$\mathrm{Aut}(\mathcal{H}) \le \mathrm{Sym}(V)$$

$\boxed{\text{HW}}$ $|\mathrm{Aut}(\text{Fano})| = 168$

collineation: aut. of proj. plane

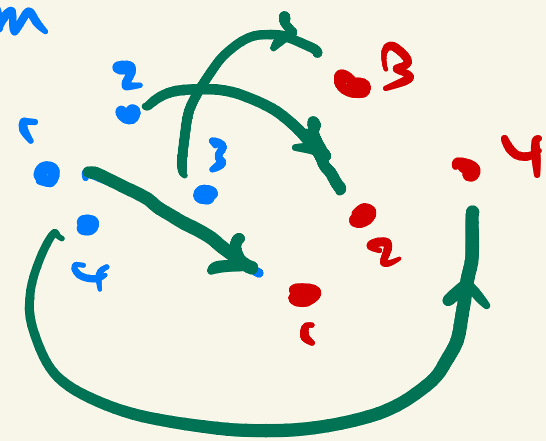
$\boxed{\text{HW}}$ $\mathbb{F}$ field $\quad PG(2, \mathbb{F})$ projective plane over $\mathbb{F}$

Fundamental Thm of Proj Geom

$\forall$ $p_1 \dots p_4$ $\quad q_1 \dots q_4$ points

no 3 on a line

$$(\exists \sigma \in \mathrm{Aut})(\forall i)(p_i^{\sigma} = q_i)$$



$G \leq \mathrm{Sym}(A)$

$x, y \in A$ $x \sim y$ if $(\exists \sigma \in G)(x^\sigma = y)$

$\boxed{\text{DO}}$ Equiv. rel.

Eq. classes: orbits orbit of $x \in A$: x^G

$$x^G = \{x^\sigma \mid \sigma \in G\}$$

$$G_x = \{\sigma \in G \mid x^\sigma = x\} \leq G$$

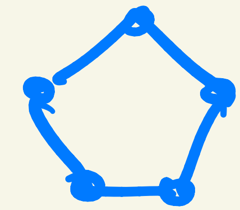
Stabilizer subgroup of x

$\boxed{\text{HW}}$ $|x^G| \cdot |G_x| = |G|$

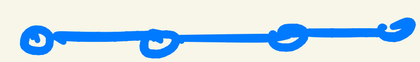
DEF G is transitive if only one orbit,

i.e. $(\forall x,y\in A)(\exists \sigma\in G)(x^{\sigma}=y)$

$\boxed{\text{DO}}$ $\mathrm{Aut}(PG(2,\mathbb{F}))$ is transitive
on points
on lines



Y



N

$\forall \mathcal{H}$ $\quad \alpha(\mathcal{H})\cdot\chi(\mathcal{H})\ge n$ $\qquad n\le \alpha\cdot\chi\le n^2$

$\boxed{\text{HW}^*}$ If $\mathcal{H}$ is vertex-transitive [Aut. transitive on V]
then $\alpha\cdot\chi\le n(1+\ln n)$

DEF random variables X, Y are <u>independent</u> if

$X, Y : \Omega \to \mathbb{R}$

$$\boxed{(\forall x, y \in \mathbb{R})\left(\Pr(X = x \wedge Y = y) = \Pr(X = x) \cdot \Pr(Y = y)\right)}$$

i.e., events "$X = x$" and "$Y = y$" are indep.

DEF $X_1 \dots X_k$ are <u>independent</u> if

$$(\forall x_1 \dots x_k \in \mathbb{R})\left(\Pr\left(\bigwedge_{i=1}^{k} X_i = x_i\right)\right) = \prod_{i=1}^{k} \Pr(X_i = x_i)$$

$\boxed{\text{HW}}$ If $\{X_i \mid i \in T\}$ indep.

then $(\forall S \subseteq T)(\{X_i \mid i \in S\})$ indep

$A_1 \dots A_k$ events

<u>atom</u> generated by the A_i:

$$A_1^{\varepsilon_1} \cap \dots \cap A_k^{\varepsilon_k} \qquad \varepsilon_i \in \{\pm 1\}$$

$$A^{1} = A \qquad A^{-1} = \bar{A}$$

2^k expressions

$\boxed{\text{DO}}$ If for all atoms Pr multiplies, i.e.

$$\Pr(A_1^{\varepsilon_1} \dots A_k^{\varepsilon_k}) = \prod \Pr(A_i^{\varepsilon_i})$$

then the A_i are indep.

<u>Thm</u> $A_1 \dots A_k$ are indep $\iff$ ϑ_{A_i} are indep.

THM If $X_1 \ldots X_n$ are indep.

$$\Rightarrow E\left(\prod_{i=1}^{n} X_i\right) = \prod_{i=1}^{n}\left(E(X_i)\right)$$

DEF X, Y are positively correlated if $E(XY) > E(X)\cdot E(Y)$

uncorrelated $=$

negatively " $<$

COROLLARY If X, Y indep $\Rightarrow$ uncorrel.

$\boxed{\text{HW}}$ $\not\Leftarrow$

$\min \leftarrow |\Omega|$

ϑ_A, ϑ_B are

pos corr $\iff$ A, B are ...

uncorr $\iff$ A, B indep

neg corr $\iff$ A, B are ...

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$$

<u>covariance</u>

$$\mathrm{Var}(X) = \mathrm{Cov}(X,X) = E\left((X-E(X))^2\right)$$

<u>variance</u>

2nd aggregate quantity for r.v.

$\boxed{\text{Ex}}$ If $X_1 \ldots X_k$ are pairwise uncorrelated

then $\mathrm{Var}(\sum X_i) = \sum \mathrm{Var}(X_i)$