COMBINATORICS PROBLEM SESSION

2024-04-19

7.64 If $q=p^k$ then $\exists\; q-1$ pairwise orthog. latin squares of order q

$$\mathbb{F}_q^{\times} = \mathbb{F}_q \setminus \{0\}$$

$$|\mathbb{F}_q^{\times}| = q-1$$

$c \in \mathbb{F}_q^{\times}$ define $L_c = \left(a_{ij}^{(c)}\right)$

rows, columns numbered by $i,j \in \mathbb{F}_q$

$$a_{ij}^{(c)} = i + cj$$

L_1 $\boxed{0\ 1 \dots}$ $\qquad$ L_2 $\boxed{0\ 2 \dots}$

no repetition: rows i

$i+cj_1 = i+cj_2 \qquad j_1=j_2$ ✓

col. j: $\quad i_1+cj = i_2+cj \qquad i_1=i_2$ ✓

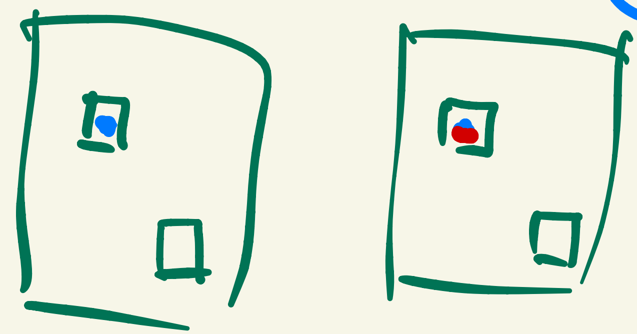
$$\begin{cases} \perp : (i_1+c_1j_1,\ i_1+c_2j_1) = \\ (i_2+c_1j_2,\ i_2+c_2j_2) \end{cases}$$

$$\underbrace{(i_1+c_1 j_1,}_{i_1 j_1}\ \underbrace{i_1+c_2 j_1)}_{i_1 j_1} = (i_2+c_1 j_2,\ i_2+c_2 j_2)$$

$$i_1 + c_1 j_1 = i_2 + c_1 j_2$$
$$i_1 + c_2 j_1 = i_2 + c_2 j_2$$

$$i_1 - i_2 = -c_1 (j_1 - j_2)$$
$$i_1 - i_2 = -c_2 (j_1 - j_2)$$

$$c_1 (j_1 - j_2) = c_2 (j_1 - j_2)$$
$$\underset{\neq 0}{(c_1 - c_2)}(j_1 - j_2) = 0$$



<u>DC</u> $i_1 = i_2$
$j_1 = j_2$

$\therefore\ j_1 = j_2$

$\therefore\ i_1 = i_2$

<u>thm</u> If $L_1 \ldots L_m$ are orthog. L.Sq's of order n then $m \le n-1$

<u>and</u> $m = n-1$ occurs $\iff \exists$ p.p. order n

$\exists$

6.15 $A_1, \dots, A_m \subseteq [n]$ $\qquad \lambda \in [n]$

s.t. $(\forall i \neq j)(|A_i \cap A_j| = \lambda)$

$\Longrightarrow$ the incidence vectors of the A_i are lin. indep. over $\mathbb{R}$ | v_i $\quad$ [∴ $m \leq n$]

method: RC Bose 1949

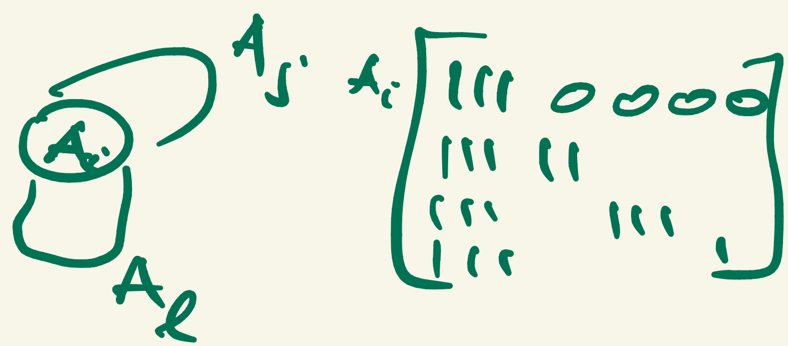
$i \neq j \qquad v_i \cdot v_j = \lambda$

Suppose $\sum a_i \underline{v}_i = \underline{0} \qquad a_i \in \mathbb{R}$

NTS: $(\forall i)(a_i = 0)$

If $(\exists i)(|A_i| = \lambda)$

✓



Now assume $(\forall i)(|A_i| > \lambda)$ $\qquad |A_i| = k_i > \lambda$

- $v_i \cdot v_j = \lambda$

$\rightarrow \quad \sum a_i \underline{v}_i = \underline{0}$

$\boxed{\text{Fisher-Rose}}$

$$0 \cdot \underline{v}_j = \left(\sum a_i v_i\right) \cdot v_j = \sum_{i=1}^{m} a_i \underbrace{v_i v_j}_{\lambda \text{ unless } i=j} = \lambda \cdot \underbrace{\sum a_i}_{S} + (k_j - \lambda) a_j$$

$$(\forall j)\left(\lambda S + \underbrace{(k_j - \lambda)}_{\neq 0} \cdot a_j = 0\right) \Leftarrow a_j = 0$$

Summing: $\Rightarrow$ $\dfrac{\lambda S}{k_j - \lambda} + a_j = 0$

$$0 = \lambda S \sum \frac{1}{k_j - \lambda} + S = S\left(\overbrace{\lambda \cdot \sum \frac{1}{k_j - \lambda} + 1}^{>0}\right) = S \cdot (>0)$$

$\therefore \underline{S = 0}$

6.23(a)

$$A = \begin{bmatrix} \alpha & & \beta \\ & \ddots & \\ \beta & & \alpha \end{bmatrix} \qquad \det A = ?$$

add every col. to last col.

$$\det : \mathbb{F}^{n\times n} \to \mathbb{F}$$

$$\det A = \det \begin{bmatrix} \alpha & \beta & \alpha+(n-1)\beta \\ \ddots & & \vdots \\ \beta & \alpha & \alpha+(n-1)\beta \end{bmatrix} = (\alpha+(n-1)\beta)\det \begin{bmatrix} \alpha & \beta & 1 \\ \ddots & & \vdots \\ \beta & \alpha & 1 \end{bmatrix}$$

$-\beta \times$ last col

$$= (\alpha+(n-1)\beta)\cdot \begin{bmatrix} \alpha-\beta & 0 & 1 \\ & \alpha-\beta & \vdots \\ 0 & \ddots & 1 \end{bmatrix} = (\alpha+(n-1)\beta)(\alpha-\beta)^{n-1}.$$

triangular matrix

6.23(b)

$$\det\begin{pmatrix}\alpha & & \beta \\ & \ddots & \\ \beta & & \alpha\end{pmatrix} = (\alpha+(n-1)\beta)(\alpha-\beta)^{n-1}.$$

$A_1 \dots A_m \subseteq [n]$ $\quad \boxed{|A_i| = k} \quad |A_i \cap A_j| = \lambda < k$

incidence matrix $M = (m_{ij})$ $\quad i = 1..m$
$m \times n$ $\quad j = 1..n$

$m_{ij} = \begin{cases} 1 & \text{if } j \in A_i \\ 0 & \text{o/w} \end{cases}$

NTS $\mathrm{rk}\, M = m$ (# rows)
"full row rank" $\quad \mathrm{rk}\, M \le m$ ✓

NTS $\mathrm{rk}\, M \ge m$

FACT $\mathrm{rk}(AB) \le \min(\mathrm{rk}\, A, \mathrm{rk}\, B)$

$$M \cdot M^T = (|A_i \cap A_j|) = \begin{pmatrix} k & & \lambda \\ & \ddots & \\ \lambda & & k \end{pmatrix}$$

$\det(MM^T) = (k+(n-1)\lambda)(k-\lambda)^{n-1} > 0$

$\mathrm{rk}(MM^T) = m$

$\mathrm{rk}\, M \ge \mathrm{rk}(MM^T) = m$ ✓
$=$

$\square$

6.26 A integral matrix rk_F $\operatorname{rk}_{\mathbb{Q}}$

Lemma rk_F only depends on $\operatorname{char} F = \begin{cases} 0 \\ p \text{ prime} \end{cases}$ $\quad \mathbb{Q} \subset \mathbb{F}$, $\mathbb{F}_p \subset \mathbb{F}$

prime fields

rk = largest r s.t. A has an $r \times r$ submatrix w. $\det_F \neq 0$

If $\operatorname{char} F = 0$ then $\det_F = \det_{\mathbb{Q}}$ ✓

$\operatorname{char} F = p$ then $\det_F = (\det_{\mathbb{Q}} \bmod p)$

$\therefore$ if $\det_{\mathbb{Q}} = 0 \implies \det_F = 0$ $\quad \therefore \operatorname{rk}_F \leq \operatorname{rk}_{\mathbb{Q}}$

Let $r = \operatorname{rk}_0(A)$ $\quad$ B subm. $r \times r$ $\quad \det_0(B) \neq 0$

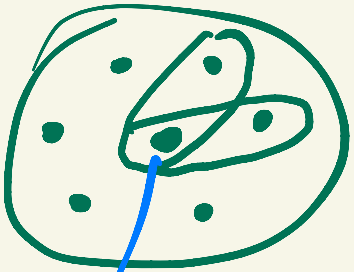
$P :=$ set of prime divisors of $\det_0(B)$

$\operatorname{char} F = p \notin P \implies \det_p(B) \neq 0 \quad \therefore \operatorname{rk}_p A \geq r$ but also $\leq r$ ✓

CH? If $A_1 \dots A_k \subseteq \Omega$ pairwise indep nontriv events, then $|\Omega| \geq k+1$

6.119 (a) $|\Omega| = k+1$ occurs $\forall k \geq 2$ (CH tight)

Ω



$\Omega = \{x_0, x_1 \cdots x_k\}$

$P(x_0) = a$

$P(x_i) = b$

$a + kb = 1$

$(a+b)^2 = a$

$(1 - kb + b)^2 = a$

$1 - 2(k-1)b + (k-1)^2 b^2 = (1-(k-1)b)^2 = 1 - kb$

nontriv $b \neq 0$

$2(k-1) + (k-1)b = -k$

$b = \frac{k-2}{(k-1)^2}$ $\quad a = \frac{1}{(k-1)^2}$ ✓

6.119 (b) $A_1 \dots A_k$ pwise indep <u>balanced</u> $\Pr(A_i) = \frac{1}{2}$

Goal: $|\Omega| \le 2k$

<u>Pf</u> $\underline{2^{r-1} < k+1} \le 2^r := |\Omega|$ $\qquad \Omega = \mathbb{F}_2^r$

$k \ge 2^{r-1}$ $\quad \therefore \quad 2k \ge 2^r$

A_i : hyperplane : $(r-1)$-dim subspace

$|A_i| = 2^{r-1} = \frac{1}{2}|\Omega|$

<u>NTS</u> $|A_i \cap A_j| = \frac{1}{4}|\Omega| = 2^{r-2}$

<u>Lemma</u> If $A, B \le V$ over any field

$\left.\begin{array}{l}\operatorname{codim} A = \operatorname{codim} B = 1 \\ A \ne B\end{array}\right\} \Longrightarrow \operatorname{codim}(A \cap B) = 2$

$m = 2^r - 1$

$A_i = a_i^\perp$

$\underline{a}_i \cdot \underline{x} = \underline{0}$

$\underline{a}_i \ne \underline{0}$

6.162 B_n n^{th} Bell number $= \#$ partitions of $[n]$

$$(*)\quad B_n = \sum_{i=0}^{n-1} \binom{n-1}{i} B_i$$

exp. gen. fctn $B(x) = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$

$$B'(x) = \sum_{n=1}^{\infty} \frac{B_n}{(n-1)!} x^{n-1} = \sum_{n=0}^{\infty} \frac{B_{n+1}}{n!} x^n$$

<u>Claim:</u> $B'(x) = e^x B(x)$

$$e^x B(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{m=0}^{\infty} \frac{B_m}{m!} x^m =$$

$n := m+k$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{B_m}{k!\,m!} x^{m+k} = \sum_{n=0}^{\infty} x^n \sum_{m=0}^{n} \underbrace{\frac{B_m}{(n-m)!\,m!}} = B'(x)$$

$$\frac{1}{n!} \sum_{m=0}^{n} B_m \binom{n}{m} = \frac{B_{n+1}}{n!}$$

ORD 6.133 $S(n,k)$ Stirling number of 2^{nd} kind

$=$ # partitions of $[n]$ into k parts

Claim: $S(n,k) \le \frac{k^n}{k!}$

$k!\,S(n,k) \le k^n$

$f : [n] \to [k]$

if f surj. then

$[n]$ $\bigcup f^{-1}(i)$ $i = 1 \dots k$

$B_n = \sum_{k=1}^{n} S(n,k)$

Dobiński's formula (1870)

$\therefore B_n < \sum_{k=0}^{\infty} \frac{k^n}{k!}$

$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}$

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7.61 If $n \geq 3$ is odd $\Longrightarrow \exists\, 2$ orthog L.Sq's of order n

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 2 & 3 & 4 & 5 & \cdots & 1 \\ 3 & 4 & 5 & \cdots & & 2 \\ \vdots & & & & & \vdots \\ n & 1 & 2 & & & n-1 \end{pmatrix}$$

$1 \; 3 \; 5 \; 7 \ldots n \; 2 \; 4 \; 6 \ldots n-1$

$1 \ldots n$

$\longrightarrow 0 \ldots n-1$

$a_{ij} = (i+j \mod n) \qquad b_{ij} = (i+2j \mod n)$

$n = \prod p_i^{k_i}$

$\Rightarrow$ by direct product

$\prod (p_i^{k_i} - 1)$ orth. L. Sq's of order n

EX Modular equation for subspaces: $A, B \le V$

$$\dim(A \cap B) + \underbrace{\dim(A+B)}_{\le r} = \underbrace{\underbrace{\dim A}_{(r-1)} + \underbrace{\dim B}_{(r-1)}}_{2r-2}$$

$$\dim(A \cap B) \ge (2r-2) - r = \underline{\underline{r-2}}$$

$\dim A = r-1$

$\dim B = r-1$

b/c $A \neq B$

$$\dim(A \cap B) < r-1$$
$$\le r-2$$

6.145 $\lambda(n) = x$ where $x \ln x = n$

Claim: $\lambda(n) \sim \frac{n}{\ln n}$

$\ln x \sim \ln x + \ln\ln x = \ln n$

$\therefore n = x \ln x \sim x \cdot \ln n$

$\lambda(n) = x \sim \frac{n}{\ln n}$