HONORS COMBINATORICS

2024-04-23

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Truncation problem

AH-HA solution by linear algebra

distinct $A_1, \dots, A_n \subseteq [n]$

$A_i^{(j)} := A_i \setminus \{j\}$ ← j^{th} truncation

Claim. $(\exists j)(A_1^{(j)}, \dots, A_n^{(j)}$ distinct$)$

Pf by lin. alg. $\mathbb{F}$ field $\mathbb{F}^n$ $e_1 \dots e_n$

standard basis

$e_i = (0 0 0 \underset{\overset{\uparrow}{i}}{1} 0 0)$

Claim. $(\exists j)(A_1^{(j)}, \dots, A_n^{(j)}$ distinct$)$

Pf by lin. alg. $\mathbb{F}$ field $\mathbb{F}^n$ $e_1 \dots e_n$ standard basis

$e_i = (0\,0\,0\,\underset{\overset{\uparrow}{i}}{1}\,0\,0)$

by contradiction

assume $(\forall j)(j^{\text{th}}$ truncation creates collision$)$

$$\Updownarrow$$

$(\forall j)(\exists i_1 \neq i_2)(A_{i_1}^{(j)} = A_{i_2}^{(j)})$ i.e. WLOG $A_{i_1} = A_{i_2} \cup \{j\}$

$j \notin A_{i_2}$

v_i: incidence vector of A_i

$\underline{e_j = v_{i_1} - v_{i_2}}$

$\mathbb{F}^n = \text{Span}(e_1 \cdots e_n) \subseteq \text{Span}(v_i - v_k \mid i,k \in [n])$

$= \text{Span}(v_1 - v_2, v_1 - v_3, \dots, v_1 - v_n)$ ← n-1 vectors span $\mathbb{F}^n$

$\rightarrow\leftarrow$ ✓

MARKOV'S INEQUALITY

$X \geq 0$ random variable

$X : \Omega \to \mathbb{R}$

$\forall a>0$

$$P(X \geq a) \leq \frac{E(X)}{a}$$

Pf

$$E(X) = \sum_{q\in\Omega} X(q)\cdot \Pr(q) \geq \sum_{\substack{q\in\Omega \\ X(q)\geq a}} X(q)\cdot \Pr(q) \geq$$

$$\geq a\cdot \underbrace{\sum_{X(q)\geq a} \Pr(q)}_{\Pr(X\geq a)} = a\cdot \Pr(X\geq a)$$

✓

X r.v.

$$\mathrm{Var}(X) = E\left((X-E(X))^2\right) \geq 0$$

variance

Chebyshev's inequality

$$\Pr\left(|X-E(X)| \geq b\right) \leq \frac{\mathrm{Var}(X)}{b^2}$$

Why not

$\underbrace{E(|X-E(X)|)}$?

nightmare

formulas

mathematical

elegance

wins

Pf. $Y = (X-E(X))^2 \geq 0$

$$\Pr\left(|X-E(X)| \geq b\right) = \Pr\left(Y \geq b^2\right) \leq \frac{E(Y)}{b^2} = \frac{\mathrm{Var}(X)}{b^2}$$

✓

DEF **Coloring** of $\mathcal{H} = (V, \mathcal{E})$ hyp.

is $f : V \longrightarrow \Sigma \longleftarrow$ "colors"

DEF **legal coloring**: no monochrom. edge

i.e. $(\forall E \in \mathcal{E})(|f(E)| \geq 2)$

DEF **chromatic number** $\chi(\mathcal{H}) = \min\{|\Sigma| : \exists$ legal coloring$\}$

\chi

$\chi(\mathcal{H}) \leq n$ ✓

3-unif: can χ be large while $(\forall E \neq E' \in \mathcal{E})(|E \cap E'| \leq 1)$?

21st century answer

SET_k : $n = 3^k$ vertices, STS

THM. $(\exists \varepsilon > 0)(\forall k)\left[\alpha\left(SET_k\right) < (3-\varepsilon)^k\right]$

$\alpha \cdot \chi \geq n = 3^k$

COR $\chi\left(SET_k\right) \geq \frac{n}{\alpha} > \left(\frac{3}{3-\varepsilon}\right)^k$

$\exists c > 0$ $= n^c$

1960s answer

PROBABILISTIC PROOF of EXISTENCE

DEF $\mathcal{H}=(V,\mathcal{E})$ almost non-intersecting if
$(\forall E\neq E'\in\mathcal{E})(|E\cap E'|\leq 1)$

THM (ERDŐS) $(\forall s)(\exists$ 3-uniform, almost non-intersecting hypergraph s.t. $\chi \geq s$

RANDOM HYPERGRAPHS

ERDŐS-RÉNYI model $\mathbb{G}^{(r)}_{n,p}$

prob. distrib. on all r-unif. hypergraphs with $V=[n]$

$$\left(\forall A\in\binom{V}{r}\right)\left(\Pr(A\in\mathcal{E})=p\right)$$

these events are indep.

GOAL:

Set $p = p_n$ (p is a function of n) s.t.

$\circledast$ $\Pr(\mathcal{H}$ is not almost non-intersecting$) \to 0$

$\circledast\circledast$ $\Pr\left(\alpha(\mathcal{H}) \geq \frac{n}{s_n}\right) \to 0$ $\quad \therefore \chi(\mathcal{H}) \geq s_n \to \infty$

Effect:

$\Pr(\circledast$ and $\circledast\circledast) \to 0$

$\therefore$ for all sufficiently large n

$\exists\, \mathcal{H}$ s.t. $\neg\circledast$ and $\neg\circledast\circledast$ ✓

TROUBLE: $\nexists$ such p

$\circledast$ forces lower bound on p

$\circledast\circledast$ forces upper bound on p

no p satisfies both

MASTERSTROKE (ERDŐS 1957)

reduce goal $\circledast$

$\circledast$ $E(\#\text{ pairs of edges } |E \cap E'| \underset{=2}{\geq 2}) \leq \frac{n}{4}$

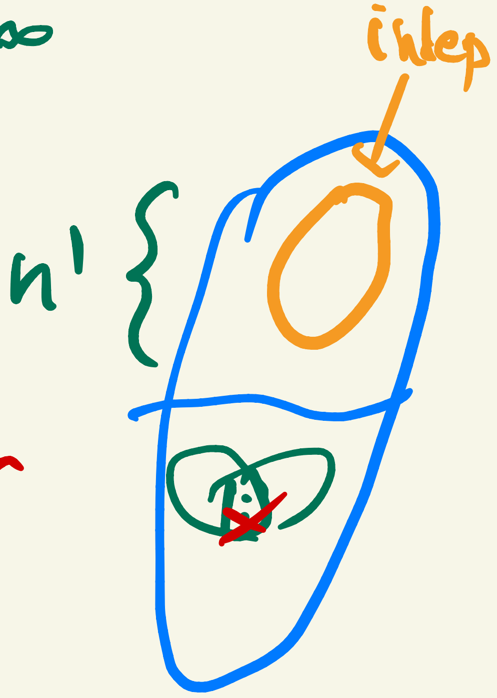
$\therefore \Pr(\#\{\text{pairs of edges } |E \cap E'| = 2\} \geq \frac{n}{2}) \leq \frac{1}{2}$ MARKOV

$\circledast\circledast$ $\Pr\left(\alpha(\mathcal{H}) > \frac{n}{s_n}\right) \longrightarrow 0$ $s_n \to \infty$

$\therefore \Pr(\circledast \text{ OR } \circledast\circledast) \lesssim \frac{1}{2}$

$\therefore (\forall n \geq n_0)(\exists \mathcal{H})(\neg\circledast \text{ and } \neg\circledast\circledast)$

Strategy: $\mathcal{H}'$: remove a vertex from each offending intersection $\therefore n' \geq \frac{n}{2}$



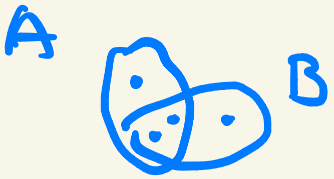
$\alpha' \leq \alpha$ $\therefore$ $\chi \geq \frac{n}{\alpha'} \geq \frac{n/2}{\alpha} \geq \frac{s_n}{2} \longrightarrow \infty$

⊛ $E(\#\text{ pairs of edges } \underset{=2}{|E\cap E'|\geq 2}) \leq \frac{n}{4}$

⊛⊛ $\Pr\left(\alpha(\mathcal{H}) > \frac{n}{s_n}\right) \longrightarrow 0$ $\qquad s_n \to \infty$

Question: calibrate p : two constraints

⊛ $\#\left\{ \underset{A\neq B}{A,B} \in \binom{V}{3} \,\middle|\, \underset{=}{|A\cap B| \geq 2} \right\}$



$E(\#\text{ bad pairs of edges}) \sim cn^4\cdot p^2$ $\qquad \binom{n}{4}\cdot 12 \sim \underline{\underline{cn^4}}$

goal: $cn^4\cdot p^2 \lesssim \frac{n}{4}$

$n^3 p^2 \lesssim c'$

$p \lesssim \frac{c''}{n^{3/2}}$

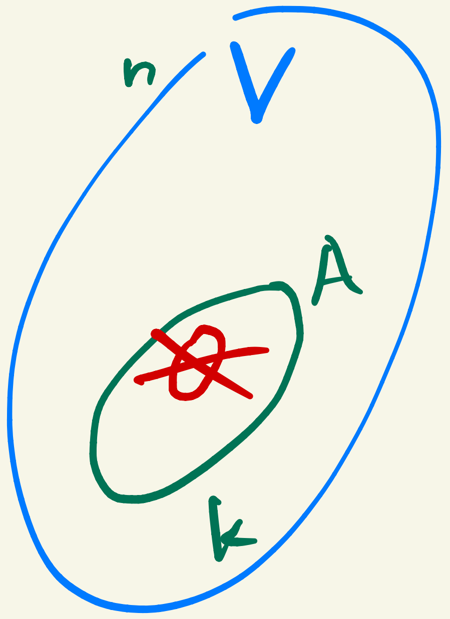
c: positive constant

NEED: $\boxed{p : \leqq \frac{c}{n^{3/2}}}$

$(\ast\ast)$

Let $A \subseteq V$, $|A| = k$

$$\Pr(A \text{ is indep}) = (1-p)^{\binom{k}{3}}$$



$$\therefore \quad \Pr(\alpha \geq k) \overset{\underline{\text{Union bound}}}{\leq} \binom{n}{k}(1-p)^{\binom{k}{3}} < \frac{n^k}{k!}\, e^{-p\binom{k}{3}}$$

$\boxed{\text{DO}}$ $\binom{n}{k} < \frac{n^k}{k!}$

$\boxed{\text{DO}}$ $(\forall x \in \mathbb{R})(1 + x \leq e^x)$

$$\Pr(\alpha \ge k) < \frac{n^k}{k!} e^{-p\binom{k}{3}}$$

$$= \frac{1}{k!}\left[n e^{-p\binom{k}{3}/k} \right]^k$$

$k = k_n \to \infty$ so for $\Pr(\alpha \ge k_n) \to 0$ it suffices

if

$$n e^{-p\binom{k}{3}/k} \le 1$$

$$n \le e^{p\binom{k}{3}/k}$$

$$\ln n \le p\binom{k}{3}/k \sim \frac{pk^2}{6} \quad \Bigg| \quad k_n = \frac{n}{s_n}$$

NEED

$$p \gtrsim \frac{6 \ln n}{k_n^2} = 6 \ln n \cdot \frac{s_n^2}{n^2} := \boxed{\frac{6 \ln n}{n^{2-2\varepsilon}}} \qquad s_n := n^{\varepsilon}$$

SUMMARIZING our two constraints on p :

$$\frac{6 \ln n}{n^{2\varepsilon}} \lesssim p \lesssim \frac{c}{n^{3/2}}$$

choose $\underline{\varepsilon < \frac{1}{4}} \implies \exists p$ ✓

Resulting chromatic number

$$\chi \geq \frac{s_n}{2} \geq \underset{\substack{\uparrow \\ \text{LARGE !}}}{\frac{n^{\varepsilon}}{2}} \quad \text{for any } \underline{\varepsilon < \frac{1}{4}}$$