$A_1, \dots, A_m \subseteq \Omega$

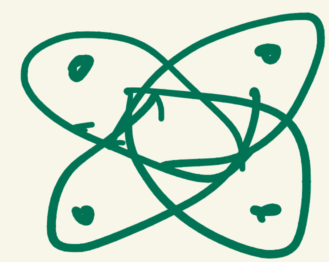
$|\Omega| = n$

the $A_i$ are not necessarily distinct

SYSTEM OF DISTINCT RERSENTATIVES

**SDR**

$x_i \in A_i \qquad i \neq j \Rightarrow x_i \neq x_j$



possible obstacles:

$\exists i \quad A_i = \emptyset$ X

$\boxed{m > n}$

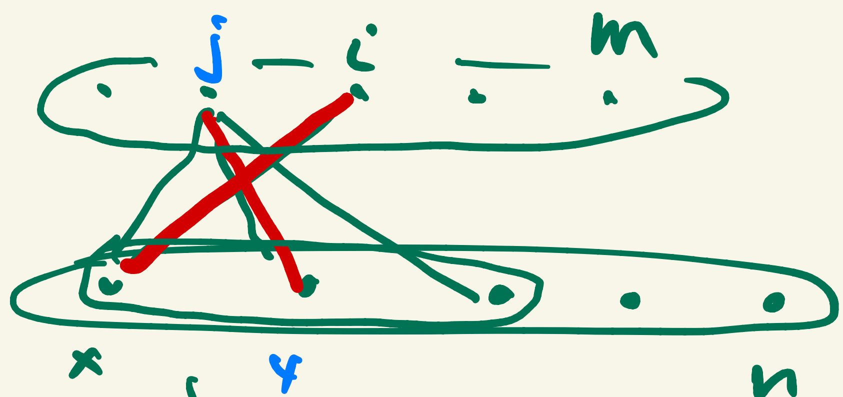
$\exists I \subseteq [m]$ s.t.

$$\left|\bigcup_{i \in I} A_i\right| < |I| \Rightarrow \text{NO SDR}$$

Philip Hall's SDR Thm / marriage thm 1935

$$\exists\, \text{SDR} \iff (\forall I \subseteq [m])\left(\left|\bigcup_{i\in I} A_i\right| \geq |I|\right)$$

"Good characterization" of $\exists$ SDR




$x \in A_i$

$y \in A_j$

DÉNES KŐNIG 1931

For bipartite graphs

$\tau = \nu$

KARL MENGER 1928

goal: $\nu = m$

KŐNIG 1916

regular bipartite graph of deg $\geq 1$

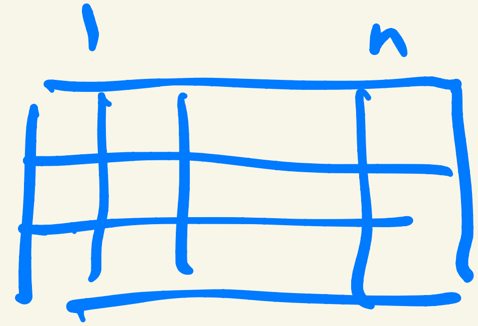
$\Longrightarrow$ has perfect matching $\quad \nu = \frac{n}{2}$

i.e. If $\mathcal{H} = (V, \mathcal{E})$ r-regular
r-uniform
hypergraph

$\Longrightarrow \exists$ SDR

MARSHALL HALL, jr : $\exists \geq r!$ perfect matchings
SDR's

# Latin Rectangle




← every row

filled with [n]

no repetition in any row or column

∀ Latin rectangle can be completed to a Latin square $\Leftarrow$ KÖNIG 1916

---

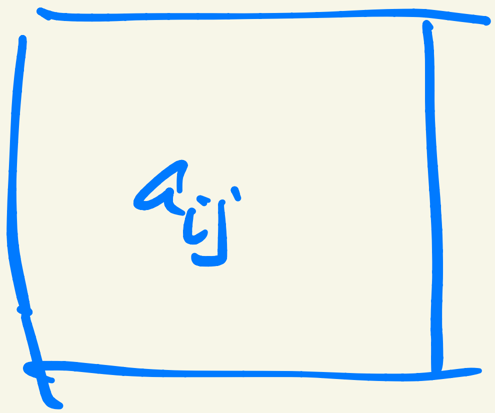
$L(n)$ : # $n \times n$ Latin squares

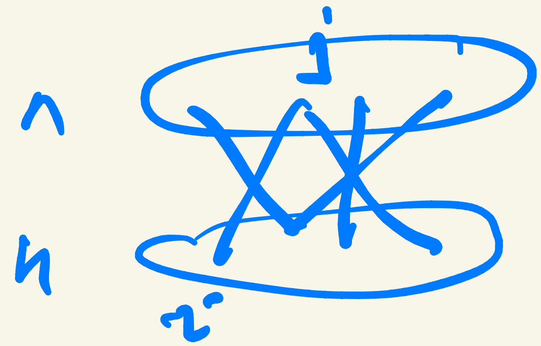
$$\prod_{k=1}^{n} k! < L(n) < (n!)^n < n^{n^2}$$

$\boxed{\text{HW}}$ $c\, n^2 \ln n \lesssim \ln L(n) < n^2 \ln n$

$c = ?$

$n \times n$




bipartite adjacency matrix

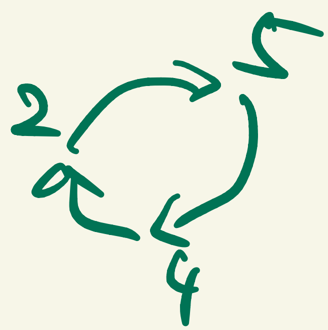
$$a_{ij} = \begin{cases} 1 & \text{if } i \sim j \\ 0 & \text{o/w} \end{cases}$$

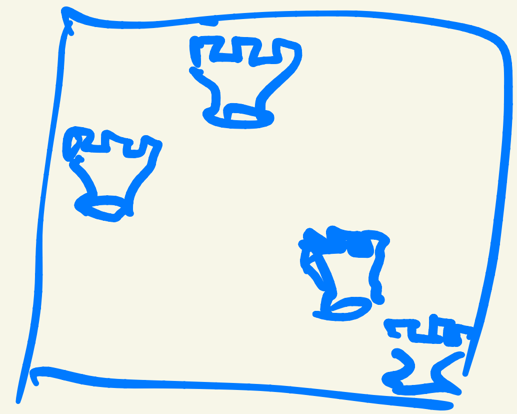
$A = (a_{ij})$

odd : $-1$
even : $+1$

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \cdot \prod a_{i\, i^\sigma}$$

$(2\ 5\ 4)$
$= (4\ 2\ 5)$



cycle of length 2:
transposition

$\boxed{\text{Do}}$ transpositions generate $S_n$

$1 \mapsto 2$
$2 \mapsto 1$
$3 \mapsto 3$
$4 \mapsto 4$

"rook configuration"

Alternating group: {even permutations}

# PERMANENT

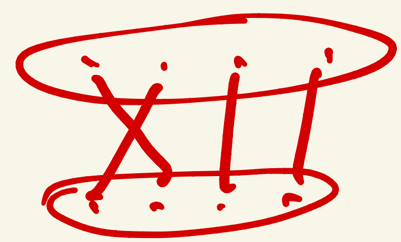
$$\operatorname{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\,i\sigma}$$

$$\begin{bmatrix} & 1 & & \\ 1 & & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$A$

bipartite adj. matrix

$\operatorname{per} A =$ # perfect matchings



Leslie Valiant

#P-complete

$n \times n$ real matrix is <u>stochastic</u>

if every row is a prob. distr:

$$a_{ij} \geq 0$$

$$\forall i \qquad \sum_{j=1}^{n} a_{ij} = 1$$

---

doubly stochastic: both $A$ and $A^{tr}$ are stoch.

$$\stackrel{\wedge}{=} \qquad \max_{\substack{A \text{ doubly} \\ \text{stoch.}}} \operatorname{per} A = 1$$

$\boxed{\text{HW}}$

$\operatorname{per} A \leq 1$

$\operatorname{per} A = 1 \iff A$ is a permutation matrix

$$\begin{bmatrix} & 1 & & & \\ & & & & 1 \\ & & & 1 & \\ 1 & & & & \\ & & 1 & & \end{bmatrix}$$

Guess $\min \operatorname{per} A$

EX $\operatorname{per} A > 0$

$J = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \end{pmatrix}$

van der Waerden's conj

$\to \operatorname{per}\left(\frac{1}{n} J\right) = \frac{n!}{n^n} > e^{-n}$

1976 EGORYCHEV – FALIKMAN

"Permanent Inequality"

$\therefore$ $r$-reg. bipartite graph $A$ row sums $r$

col " $r$

$\frac{1}{r} A$

$\boxed{\text{HW}}$ Prove: $\ln L(n) \sim n^2 \ln n$

# Fisher - Bose thm Majumdar

$A_1, \dots, A_m \subseteq [n]$

$\exists \lambda \geq 1 \quad (\forall i \neq j)(|A_i \cap A_j| = \lambda)$

$\Longrightarrow m \leq n$ b/c incid. vectors lin indep

---

$M$ incid matrix

$$MM^{tr} = \begin{pmatrix} k_1 & & \lambda \\ & \ddots & \\ \lambda & & k_m \end{pmatrix} \qquad k_i = |A_i| > \lambda$$

---

**Def** A sym real matrix : positive semidef if $\forall \underline{x} \quad x^{tr} A x \geq 0$

$$x^{tr} A x = \sum_{i,j} a_{ij} x_i x_j$$

pos. definite $> 0 \quad x \neq 0$

Ex Lemma $J$ is pos. semidef

Lemma $\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}$ $d_i > 0$ pos def

---

Obs pos semi + pos def = pos def

---

Obs pos def $\Longrightarrow$ nonsingular ✓