PROBLEM SESSION

2024-04-26

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6.117 n balanced events $(n-1)$-wise but not fully indep.

$\Omega = \mathbb{F}_2^n \ni (x_1 \ldots x_n)$ uniform

$A_i = \{ \underline{x} \in \Omega \mid x_i = 0 \}$

$\Omega' = \{ x \in \mathbb{F}_2^n \mid \underline{x} \perp \underline{\mathbb{1}} \}$ $\quad \sum x_i = 0$

$|\Omega'| = 2^{n-1}$ NTS. $I \subsetneq [n] \Rightarrow P\left(\bigcap_{i \in I} A_i\right) = \frac{1}{2^{|I|}}$

this is opt. b/c $\quad |I| = k$

If $\exists k$ non-triv. indep. events then $|\Omega| \geq 2^k$

Bell numbers B_n : # partitions of $[n]$

$$B_n < n^n$$

$$\underbrace{(\forall \varepsilon)(\exists n_\varepsilon)(\forall n > n_\varepsilon)}_{\text{all sufficiently large } n}\left(B_n < (\varepsilon n)^n\right)$$

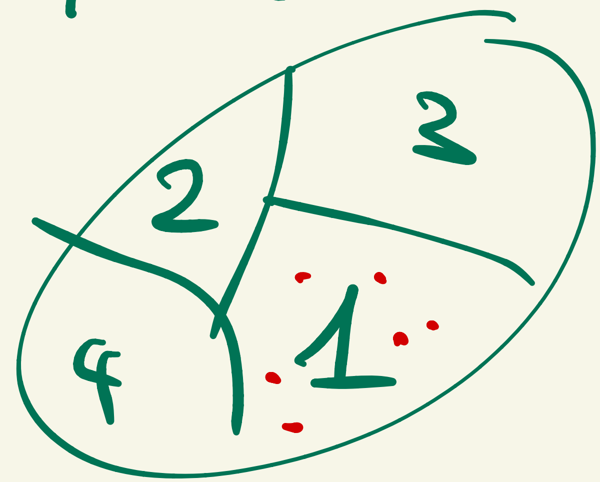
$$B_n = \sum_{k=1}^{n} S(n,k)$$

$$\boxed{S(n,k) \leq \frac{k^n}{k!}}$$

$S(n,k)$: # partitions of n into k parts

$$S(n,k) \le \frac{k^n}{k!}$$

$k!\,S(n,k)$: # ordered partitions $\hookrightarrow [k]^{[n]}$
into k parts



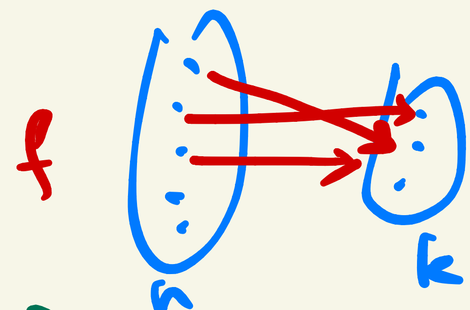
Claim

$\underline{k!\,S(n,k)} \le k^n$

injective proof

k^n counts $f : [n] \to [k]$

why is this map not surjection:



$\underline{F}$(ordered partition X) $= \underline{f} : [n] \to [k]$

$$B_n = \sum_{k=1}^{n} S(n,k) < \sum_{k=1}^{n} \frac{k^n}{k!} < \sum_{k=1}^{n} k^n$$

$$n^n$$

<u>NTS</u> $B_n < (\underline{\varepsilon n})^n$

$$\frac{k^n}{k!} = \underbrace{\frac{k^k}{k!}}_{<e^k} \cdot k^{n-k} < e^k \cdot k^{n-k}$$

$$k! > \left(\frac{k}{e}\right)^k$$

$$e^n = \sum_{j=0}^{\infty} \frac{n^j}{j!} > \frac{n^n}{n!} \qquad \therefore \; n! > \frac{n^n}{e^n}$$

NTS: $B_n < (\varepsilon n)^n$

$$B_n < \sum_{k=1}^{n} \frac{k^n}{k!} < \sum_{k=1}^{n} e^k \cdot k^{n-k}$$

$$\frac{k/n}{\underbrace{\left(k/e\right)}_{\to\infty}^{\underbrace{k/n}_{\geq \varepsilon}}} \overset{?}{<} \varepsilon \qquad < 1$$

if $k < \delta n$

$$\Rightarrow e^k k^{n-k} < e^k (\delta n)^n < (\underbrace{\delta e}\, n)^n < (\varepsilon' n)^n$$

$\delta e < \varepsilon$

$\delta : < \frac{\varepsilon}{e} = \varepsilon'$

$$\therefore \sum_{k < \varepsilon n} \frac{k^n}{k!} < \frac{(\varepsilon n)^n}{100}$$

$k > \varepsilon n$ $\qquad e^k \cdot k^{n-k} = \dfrac{k^n}{\left(\frac{1}{e} k\right)^k}$

$$\frac{k^n}{(k/e)^k} < (\varepsilon n)^n$$

$$\frac{k}{(k/e)^{k/n}} \overset{?}{<} \varepsilon n$$

$n \times n$ cyclic Latin Square $\boxed{a_{ij} = i - j \bmod n}$

If n is even
$\Rightarrow$ no orthogonal mate

$$\begin{pmatrix} a_0 & \boxed{a_1} & \cdots & a_{n-1} \\ \boxed{a_{n-1}} & a_0 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & \cdots & a_{n-3} \\ \vdots & & & \end{pmatrix}$$

$\Sigma = \{0, \ldots, n-1\}$

$$\sum \Sigma = \sum_{j=0}^{n-1} j = \frac{n(n-1)}{2}$$

(b_{ij})

Transversal: rook configuration with all a_{ij} distinct

Claim $\#$ —"— : $\sum a_{i,i^\sigma} = \frac{n(n-1)}{2} = \binom{n}{2}$

$\binom{n}{2} = \sum a_{i,i^\sigma} = \sum i - i^\sigma \bmod n = \binom{n}{2} - \binom{n}{2} = 0 \bmod n$

i.e. $n \mid \frac{n(n-1)}{2}$

$\therefore \frac{n-1}{2} \in \mathbb{Z}$

CH circulant matrix

$$C(a_0, \dots, a_{n-1}) = (a_{ij}) \qquad a_{ij} = i-j \bmod n$$

$\lambda_0, \dots, \lambda_{n-1}$: eigenvalues

Claim $f(x) = \sum_{j=0}^{n-1} a_j x^j$

ω : primitive nth root of unity $\omega^n = 1$

for $0 < k < n$

$\omega^k \neq 1$

$$\boxed{\lambda_j = f(\omega^j)}$$

$\therefore \det C = \prod \lambda_j$

Littlewood-Offord problem

$a_1, \dots, a_n, b \in \mathbb{R}, \quad a_i \neq 0$

good subset : $I \subseteq [n]$ s.t. $\sum_{i \in I} a_i = b$

$$\Pr(I \subseteq [n] \text{ is good}) = O\left(\frac{1}{\sqrt{n}}\right)$$

$$(\exists a_1, \dots a_n, b)(P \quad) = \Theta\left(\frac{1}{\sqrt{n}}\right)$$

assume $2 | n$

$a_1 = a_2 = \dots = a_n = 1 \qquad b = \frac{n}{2}$

good subsets $\binom{n}{n/2}$ $\qquad \Pr = \frac{\binom{n}{n/2}}{2^n} \sim \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{n}}$

Assume ~~WLOG~~ that $(\forall i)(a_i > 0)$

$\mathcal{F} = \{ I \mid I \text{ is good} \}$ $\qquad \sum_{i \in I} a_i = b$

$\mathcal{F}$ is a Sperner family

i.e. if $I, J \subseteq [n]$ good then $I \nparallel J$

$I \not\subseteq J$

$J \not\subseteq I$

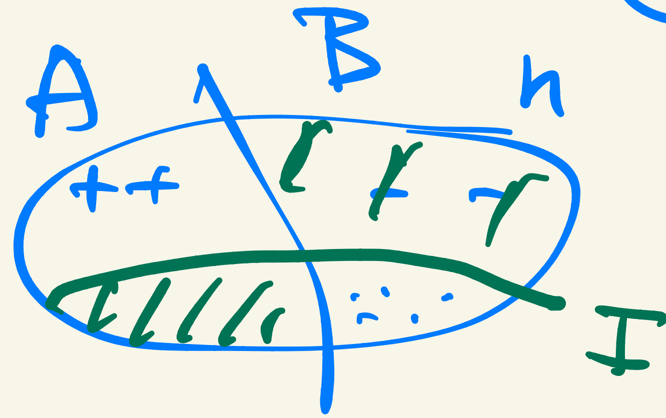
$$\underbrace{\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot}_{b} \quad \cdot \quad \cdot$$

$\therefore \ |\mathcal{F}| \le \binom{n}{n/2}$

$[n] = A \,\dot\cup\, B$

$i \in A \Rightarrow a_i > 0$

$i \in B \Rightarrow a_i < 0$



I

$$\sum_{i \in \underline{I \cap A}} a_i - \sum_{j \in \underbrace{\underline{B \setminus I}}_{T - \sum_{j \in I \cap B} a_j}} a_j =$$

$$= \underbrace{\sum_{i \in I \cap A} a_i + \sum_{j \in I \cap B} a_j}_{\sum_{i \in I} a_i = b} - T = b - T$$

$\sum_{i \in A} a_i = S$

$\sum_{j \in B} a_j = T$

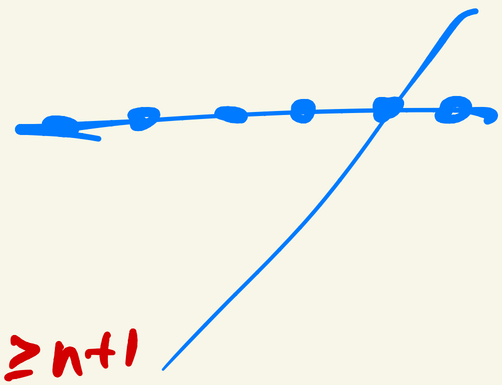
$\tilde{\mathcal{F}} = \{ (I \cap A) \cup (B \setminus I) \mid I \text{ good} \}$

$\tilde{\mathcal{F}}$ Sperner

$\mathcal{P}(P, L, I)$ proj. plane of order n

Q $\tau(\mathcal{P})$ covering number (hitting number)

$\tau \le n+1$ take any line

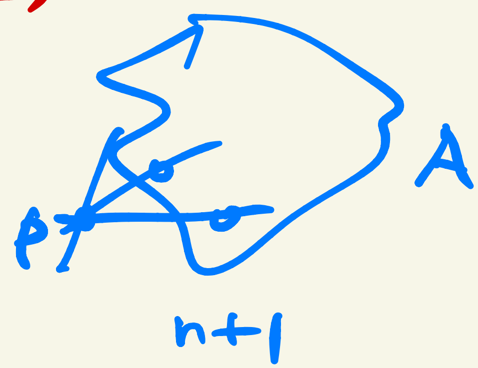


$\tau \ge n+1$?

~~Let $A \subseteq \mathcal{P}$ $|A| \le n$~~

Claim A is ~~NOT~~ a cover $\Rightarrow$ $|A| \ge n+1$

pick $p \notin A$



Club of 2000

$$|\Omega| = 2000!$$

$X = \#$ lucky members

$$X = \sum_{i=1}^{2000} Y_i$$

Y_i indicates "i^{th} member is lucky"

$$E(X) = \sum E(Y_i) = \sum P(\quad) = \sum_{i}^{n} \frac{1}{n} = 1.$$

Z_i indicates "i^{th} card is lucky"

$$X = \sum Z_i$$

$$P(\quad) = \frac{n_i}{2000}$$

n_i : # members born in year i

$$E(X) = \sum E(Z_i) = \sum_{i=1}^{2000} \frac{n_i}{2000} = 1$$

$$\sum n_i = 2000$$

def τ^* is sound

$\underline{x} = (x_1, \dots, x_n) \quad x_i \in \mathbb{R}$

$$\tau^* = \underbrace{\min}_{\text{inf}} \left\{ \sum x_i \;\middle|\; x_i \geq 0 \; (\forall E \in \mathcal{E}) \left(\sum_{i \in E} x_i \geq 1 \right) \right\}$$

$\sum x_i \leq n$

Extreme value thm

$$\left.\begin{array}{l} f : C \to \mathbb{R} \\ \text{continuous} \end{array}\right\} \Rightarrow \min f \text{ is attained}$$

$C \subseteq \mathbb{R}^n$

compact: bounded, closed

$C \neq \emptyset$

9.52 r-unif $m \le 2^{r-1}$ edges $\Rightarrow$ 2-colorable

First assume $m < 2^{r-1}$ $\{A_1 \dots A_m\} = \mathcal{E}$

$Pr(A_i \text{ monochrom.}) = \frac{2}{2^r} = \frac{1}{2^{r-1}}$

random 2-coloring

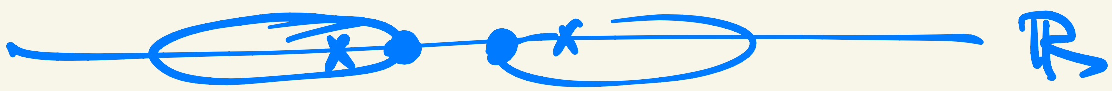
UNION BOUND

$Pr((\exists i)(A_i \text{ mono.})) \le \frac{m}{2^{r-1}}$

= in union bound means the events are (almost) disjoint

Claim: if $m \ge 2 \Rightarrow$ these events are not disj.: color $\forall$ vertex "red"

9.139 $\nu^* \leq \tau^*$



<u>Claim</u> value any fractional matching $\leq$ value of any fract. cover