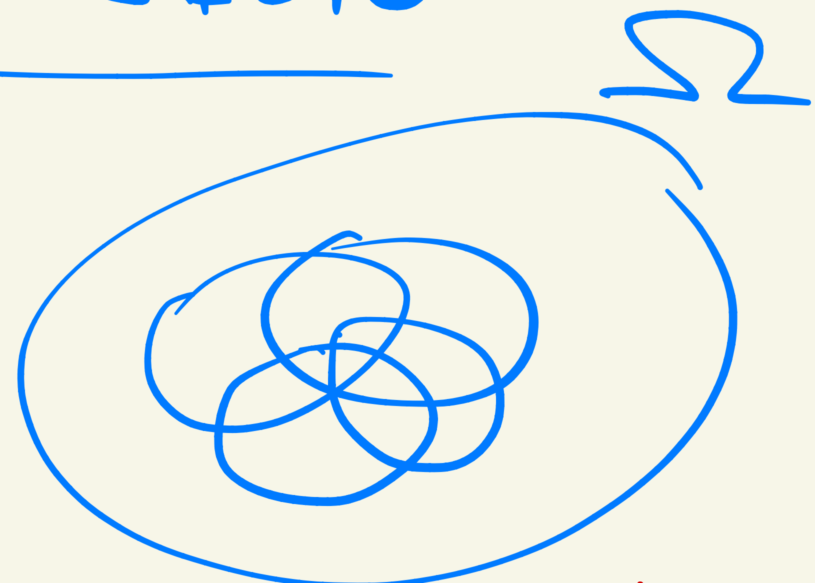
HONORS COMBINATORICS

2024-04-30

INCLUSION-EXCLUSION



$A_1, \dots, A_k \subseteq \Omega$

$B = \overline{\bigcup_{i=1}^{k} A_i}$

Given for all $I \subseteq [k]$ $\quad P\left(\bigcap_{i \in I} A_i\right)$ $\qquad 2^k$ data

find $P(B) = S_0 - S_1 + S_2 - + \dots + (-1)^k S_k$

where

$$S_j = \sum_{\substack{I \subseteq [k] \\ |I| = j}} P\left(\bigcap_{i \in I} A_i\right)$$

$I = \emptyset$

$\bigcap_{i \in I} A_i = \Omega$

$S_0 = 1$

$S_1 = \sum P(A_i)$

$S_2 = \sum P(A_i \cap A_j) \qquad i < j$

$$B = \overline{\bigcup_{i\in[k]} A_i}$$

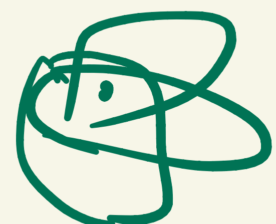
$$\Phi := S_0 - S_1 + S_2 - + \cdots$$

$$= \sum_{a\in\Omega} \beta_a \cdot \Pr(a)$$

Claim: $\beta_a = \begin{cases} 1 & \text{if } a \in B \\ 0 & \text{o/w} \end{cases}$

$\underline{a}$ is counted in S_j $\binom{d(a)}{j}$ times

$d(a) = \deg(a)$



$$\beta_a = \binom{d(a)}{0} - \binom{d(a)}{1} + \binom{d(a)}{2} - + \cdots$$

$$= (1-1)^{d(a)}$$

$$= 0^{d(a)} = \begin{cases} 1 & \text{if } d(a) = 0 \\ 0 & \text{if } d(a) > 0 \end{cases}$$

i.e. $a \in B$

$a \notin B$

Binomial Thm

$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j$$

Proof #2 $\quad \vartheta_i$: indicator of A_i

$$A_i \cap A_j : \vartheta_i \cdot \vartheta_j$$

$$\bar{A}_i = 1 - \vartheta_i$$

$$\vartheta_B = \vartheta_{\cap \bar{A}_i} = \prod \vartheta_{\bar{A}_i} = \prod_{i=1}^{k} (1 - \vartheta_i)$$

$$= \sum_{I \subseteq [k]} (-1)^{|I|} \prod_{i \in I} \vartheta_i = \sum_{I \subseteq [k]} (-1)^{|I|} \vartheta_{\bigcap_{i \in I} A_i}$$

$$\boxed{\prod_{i=1}^{k} (1 + x_i) = \sum_{I \subseteq [k]} \prod_{i \in I} x_i}$$

$$\therefore E(\vartheta_B) = \sum (-1)^{|I|} E(\vartheta_{\cap A_i})$$

$$\Pr(B) = \sum_{I \subseteq [k]} (-1)^{|I|} \Pr\Big(\bigcap_{i \in I} A_i\Big)$$

Bonferroni's Inequalities

$$Pr(B) \leq S_0 = 1$$

$$Pr(B) \geq S_0 - S_1$$

$$Pr(B) \leq S_0 - S_1 + S_2$$

$$\vdots$$

$$Pr(B) \underset{\leq}{\geq} S_0 - S_1 + \cdots \pm S_j$$

— if j odd

— if j even

What if input $\Pr\left(\bigcap_{i \in |I|} A_i\right)$ for $|I| \le \ell$

$\Pr\left(\bigcup_{i=1}^{k} A_i\right) \approx ?$ if ℓ

NOAM NATI

NISAN-LINIAL ~1990

$\log k$ ←

$\boxed{\sqrt{k}}$ ←

$k^{1-\varepsilon}$

ck

LP duality

approximation theory

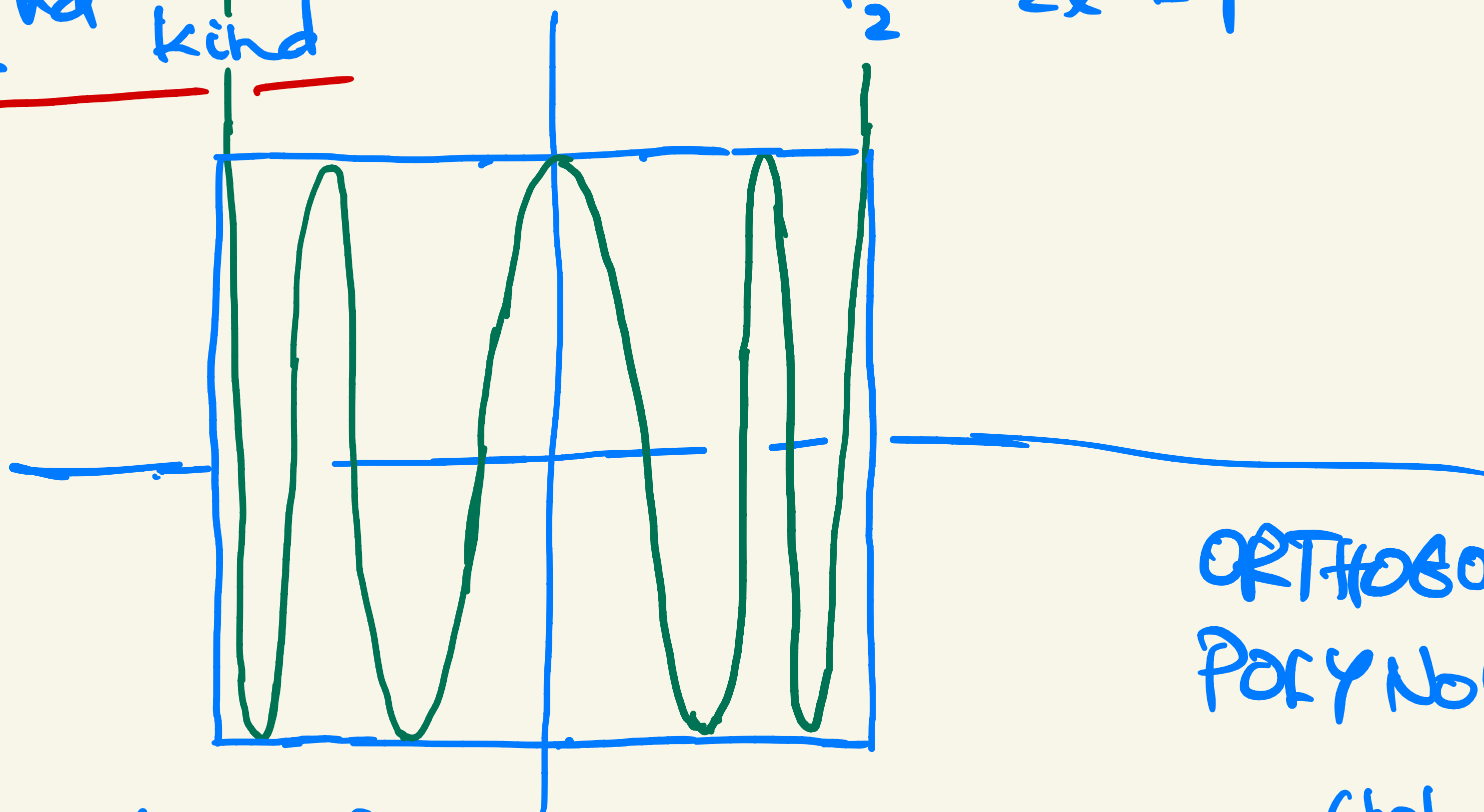
Chebyshev polynomials

$$\cos(n\vartheta) = T_n(\cos\vartheta)$$

$$\cos 2\vartheta = 2\cos^2\vartheta - 1$$

$$T_2 = 2x^2 - 1$$

Chebyshev polynomials of 2^{nd} kind



ORTHOGONAL POLYNOMIALS

$$\frac{\sin(n+1)\vartheta}{\sin\vartheta} = U_n(\cos\vartheta)$$

← Chebyshev polynomials of the 1^{st} kind

$$A_1 \dots A_m \subseteq [n]$$

$$|A_i \cap A_j| = \lambda$$

$\Longrightarrow m \le n$

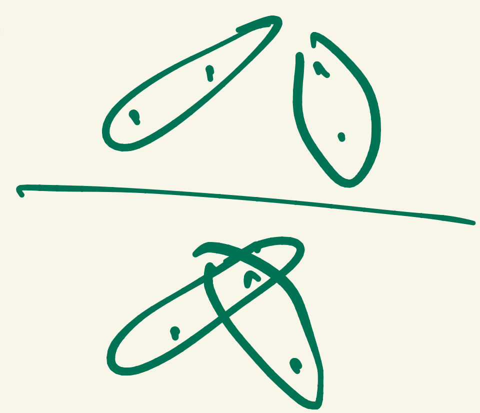
incid. vectors lin indep.

(*) $|A_i \cap A_j| \in \{\lambda, \mu\}$

$|A_i| = k$ uniform

max m (*) in terms on of n

$\binom{n}{2}$ $\qquad \mathcal{F} = \binom{[n]}{2}$ $\qquad \lambda = 0$, $\mu = 1$



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$L = \{\ell_1, \dots, \ell_s\}$

$\mathcal{F} = \{A_1, \dots, A_m\}$ is L-intersecting if

$(\forall i \neq j)(|A_i \cap A_j| \in L)$

uniform case: $(\forall i)(|A_i| = k)$

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max m : $\geq \binom{n}{s}$ take $\mathcal{F} = \binom{[n]}{s}$

$s \leq \frac{n}{2}$

THM DIJEN K. RAY-CHAUDHURI – RICHARD M. WILSON

1964

max $m = \binom{n}{s}$

DEF s-inclusion matrix

$$M = \begin{array}{c|ccc} & B_1 & \cdots & B_{\binom{n}{s}} \\ \hline A_1 & & & \\ \vdots & & \left(m_{ij}^{(s)}\right) & \\ A_m & & & \end{array}$$

columns labeled by $\binom{[n]}{s}$

$B_i \subseteq [n]$

$|B_i| = s$

RW: MM^{tr} nonsingular

$$m_{ij}^{(s)} = \begin{cases} 1 & \text{if } B_j \subseteq A_i \\ 0 & \text{o/w} \end{cases}$$

$$MM^{tr} = \left(c_{it} \right)_{m \times m} \qquad c_{it} = \binom{|A_i \cap A_j|}{s}$$

$i, t \in [m]$

non-uniform R-W inequality

$A_1 \ldots A_m \subseteq [n]$ L-intersecting

PETER FRANKL - WILSON

$$\Longrightarrow \quad m \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}$$

$\underline{x},\underline{y}\in\mathbb{R}^n$ $\qquad |A_i|=k > \ell_j$

$$F(\underline{x},\underline{y})=\prod_{j=1}^{s}(\underline{x}\cdot\underline{y}-\ell_j)$$

$$F(A_i,A_t)=\begin{cases}\prod_j(k-\ell_j)\neq 0 & i=t\\ 0 & i\neq t\end{cases}$$

$$\prod_j(|A_i\cap A_t|-\ell_j)$$

$$\big(F(A_i,A_t)\big)=\begin{Bmatrix}\lambda & & 0\\ & \ddots & \\ 0 & & \lambda\end{Bmatrix}$$

incid. vector of A_t

$f_t(\underline{x}):=F(\underline{x},\underline{z}_t)$

$f_1,\dots,f_m$

Claim: the f_i are lin indep.

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vars $\quad$ $\deg \le s$ $\quad$ monomial

$$d(n,s) := \mathrm{Poly}(n, s) = \mathrm{span}\left(\prod x_i^{k_i}\right)$$

deg $\sum k_i \le s$

$d(n,1) := n+1$ $\qquad$ $\mathrm{span}(x_1 \dots x_n \; 1)$

$d(n,2) := \binom{n+2}{2}$ $\qquad$ $\mathrm{span}(x_i^2, x_i x_j, x_i, 1)$

$$\begin{array}{cccc} x_i^2 & x_i x_j & x_i & 1 \\ n & \binom{n}{2} & n & 1 \end{array}$$

$$2n + \frac{n(n-1)}{2} + 1 =$$

$$= \frac{n^2+3n+2}{2} = \binom{n+2}{2}$$