field $\mathbb{F}$ $\qquad$ $\mathbb{F}[x] = \{ f \mid \text{polynomial over } \mathbb{F} \}$

coefficients from $\mathbb{F}$

$f(x) = a_0 + a_1 x + \dots + a_n x^n + 0 x^{n+1} + \dots$

$(a_0, a_1, \dots)$ all but a finite number of coeff's $= 0$

$1 = (1, 0, 0, \dots)$
$x = (0, 1, 0, \dots)$
$x^2 = (0, 0, 1, \dots)$

$1, x, x^2, \dots$: basis of $\mathbb{F}[x]$

algebra: vector ~~space~~ ring

pseudo-associativity: $\alpha(ab) = (\alpha a)b = a(\alpha b)$ $\quad \alpha \in \mathbb{F}$, $a, b \in \mathbb{F}[x]$

$\mathbb{F}$ valued functions
functions
with domain Ω

$\mathbb{F}^\Omega = \{ f : \Omega \to \mathbb{F} \}$

algebra: ptwise op's

$T : \mathbb{F}[x] \to \mathbb{F}^{\mathbb{F}}$ $\qquad f \longmapsto \tilde{f}$

$f \longmapsto f(x) \leftarrow$ function $\tilde{f}$

formal expression $\qquad$ plug in $x \in \mathbb{F}$

EX. T is injective $\iff$ $\mathbb{F}$ is infinite

necessity $\Longrightarrow$ suppose $\mathbb{F} = \{\alpha_0, \alpha_1, \dots, \alpha_{q-1}\}$

$f(x) = \prod_{i=0}^{q-1} (x - \alpha_i) \qquad \tilde{f} = 0$ function

- $\deg(f) = \max\{k : \operatorname{coeff}(x^k) \neq 0\}$

$\deg(f) = 0 \iff f$ is a nonzero constant: $f = a_0 \neq 0$

$\deg(0)$

---

(1) $\deg(f+g) \leq \max(\deg f, \deg g)$

(2) $\deg(f \cdot g) = \deg f + \deg g$

If extends to 0 poly:

(1) $\deg(0) = \deg(f + (-f)) \leq \max(\deg f, \deg(-f)) = \deg f$

$\therefore (\forall f)(\deg(0) \leq \deg(f))$ $\therefore \deg(0) \leq 0$

$\deg(0) = -\infty$

(2) $\underbrace{\deg(0 \cdot f)} = \deg(0) + \deg(f)$

$\deg(0) = \deg(0) + k$ $\rightarrow\leftarrow$

$\deg(0) = \pm\infty$

polynomials in n variables $\mathbb{F}[x_1, \dots x_n]$

<u>basis</u>: monomials

$$m(x) = x_1^{k_1} \dots x_n^{k_n} \qquad k_i \in \mathbb{N}_0$$

$$3x_1^5 x_7 - \frac{1}{4} \cdot x_1^6 x_3^5$$

<u>$\deg m = \sum k_i$</u>

$\deg f = \max \deg$ (monomials involved)

$\deg 0$ (empty lin comb) $:= -\infty$

---

(1), (2) hold

$$\mathbb{F}^{\le k}[x_1, \dots x_n] = \{ f \in \mathbb{F}[x_1 \dots x_n] \mid \deg f \le k \}$$

subspace

HW: dim ?

RAY-CHAUDHURI - WILSON THM

$L = \{\ell_1, \ldots, \ell_s\} \subset \mathbb{N}_0$

DEF $\mathcal{A} = (A_1, \ldots, A_m)$ is *L-intersecting*
if $(\forall i \neq j)(|A_i \cap A_j| \in L)$

THM If $\mathcal{A} \subseteq \binom{[n]}{k}$ k-uniform hypergraph with n vertices
L-intersecting where $s \leq k$
then

$$m \leq \binom{n}{s}$$

Tight given n, s:

$\mathcal{A} = \binom{[n]}{s}$ $L = \{0, \ldots, s-1\}$

NON-UNIFORM R-W : FRANKL-WILSON

If $\mathcal{A} \subseteq \mathcal{P}([n])$, L-intersecting

then $$m \le \binom{n}{s} + \binom{n}{s-1} + \dots + \binom{n}{0}$$

Given n, s this is tight:

take all subsets of size $\le s$

$T \in \binom{[n]}{s}$

$$\begin{array}{c|c}
 & T \\
\hline
A_1 & \\
\vdots & 1 \\
A_m & \\
\end{array}$$

1 if $T \subseteq A_i$

Original proofs based on higher incidence matrices

specifically: inclusion matrices

edges vs subsets of size $\begin{cases} s & \text{RW} \\ \le s & \text{FW} \end{cases}$

Lemma If $f_1, \dots, f_m \in \mathbb{F}[x_1 \dots x_n]$

$a_1, \dots, a_m \in \mathbb{F}^n$

and

$$f_i(a_j) = \begin{cases} \neq 0 & i = j \\ 0 & i \neq j \end{cases}$$

Then the f_i are lin. indep.

Pf Suppose $h = \sum \alpha_i f_i = 0$ fctn $\alpha_i \in \mathbb{F}$

NTS $(\forall i)(\alpha_i = 0)$

$$0 = h(a_j) = \sum_i \alpha_i \underbrace{f_i(a_j)}_{} = \alpha_j \cdot \underbrace{f_j(a_j)}_{\neq 0}$$

$\alpha_j \cdot \text{nonzero} = 0$

$\forall j$ $\therefore \alpha_j = 0$

$\underline{x}, \underline{y} \in \mathbb{F}^n$ $L = \{\ell_1 \dots \ell_s\}$ (*)

$$F(\underline{x}, \underline{y}) := \prod_{t=1}^{s} (\underline{x} \cdot \underline{y} - \ell_t)$$

RW conditions

v_i : incid. vector of A_i

$$F(\underline{v}_i, \underline{v}_j) = \begin{cases} \prod (k - \ell_t) \neq 0 \\ 0 \text{ if } i \neq j \end{cases}$$

$$\underline{v}_i \cdot \underline{v}_j = |A_i \cap A_j|$$

WLOG $(\forall t)(\ell_t < k)$

$$f_i(\underline{x}) := \prod_{t=1}^{s} (\underline{x} \cdot \underline{v}_i - \ell_t)$$

$\underline{x} \in \mathbb{F}^n$

$\deg f_i = s$

$f_i \in \mathbb{F}[x_1 \dots x_n]$

$$f_i(v_j) = \begin{cases} \neq 0 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

$\therefore f_1 \dots f_m$ are lin indep

$\dim \mathbb{F}^{\leq s}[x_1, \ldots x_n]$

$s=1$ $\quad n+1$

$s=2$ $\quad \binom{n+2}{2}$

included monomials of the form f $\quad x_1^3 x_5^2 x_7$

$\downarrow$

if $x \in \{0,1\}$ then $x = x^2$

reduced polynomial $\quad \bar{f} \quad \underline{x_1 x_5 x_7}$

$\deg(\bar{f}) \leq \deg(f) = s$

$\bar{f}$: <u>multilinear</u> : linear in each variable dim(space of multilinear polynomials in n variables of degree $\leq s$) $=$

for $I \subseteq [n]$

$$x_I = \prod_{i \in I} x_i$$

basis: $\{ x_I \mid |I| \leq s \}$

$$\# \quad \underbrace{\overbrace{\binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}}}_{(*)}$$

$\therefore$ Under R-W conditions

$$m \leq (*)$$

Claim: $f_1, \dots, f_m, \{ \overbrace{(\sum x_i - k)\cdot x_I}^{g_I} \mid \underbrace{|I| \le s-1}_{} \}$

are lin indep.

$$\boxtimes = \sum_{r=0}^{s-1} \binom{n}{r}$$

$$\therefore \quad m + \boxtimes \le (*) = \sum_{r=0}^{s} \binom{n}{r}$$

RW

Pf Suppose $\sum \alpha_i f_i + \sum_{|I|\le s-1} \beta_I g_I = 0$

plug in v_j: $\longrightarrow$ $g_I(v_j) = 0$

$\therefore \quad \sum \alpha_i f_i(v_j) = 0$

$\forall j \quad \therefore \quad \alpha_j = 0 \qquad \therefore \quad \sum \beta_I g_I = 0 \quad \therefore \quad \forall \beta_I = 0$ by def

Pf of F-W nonuniform

NON-UNIFORM R-W : FRANKL-WILSON

If $A \subseteq \mathcal{P}([n])$, L-intersecting

then $$m \le \binom{n}{s} + \binom{n}{s-1} + \dots + \binom{n}{0} \qquad (*)$$

Try

$$f_i(\underline{x}) = \prod_{t=1}^{s} (\underline{x} \cdot \underline{v}_i - \ell_t)$$

~~$m \le (*) + s$~~

$$f_i(v_j) = \begin{cases} ? & i=j \\ 0 & i \neq j \end{cases}$$

0 if $|A_i| \in L$

$\exists t$

$\underline{v}_i \cdot \underline{v}_j = |A_i \cap A_j| = \ell_t$

$$|A_1| \le |A_2| \le \cdots \le |A_m|$$

$$f_i(x) := \prod_{\ell_t < |A_i|} (x \cdot v_i - \ell_t) \qquad \text{so } f_i(v_i) \neq 0$$

$$f_i(v_j) = 0 \quad \text{if} \quad \underline{i > j}$$

$$\prod_{\ell_t < |A_i|} \left(|A_i \cap A_j| - \ell_t\right) \qquad \cancel{A_i \neq A_j}$$

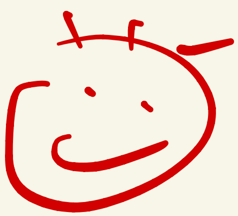
$$\therefore |A_i \cap A_j| = \ell_t < |A_i|$$

$$|A_i \cap A_j| = |A_i| \longleftarrow \times$$

$$\Rightarrow A_i \subseteq A_j \Rightarrow |A_i| = |A_j|$$

$$\Rightarrow |A_i \cap A_j| < |A_i| \longleftarrow \times$$

$$\therefore \; m \le \sum_{r=0}^{s} \binom{n}{r}$$



F-W

$f_1 \ldots f_m$ fctns

$a_1 \quad a_m \in \mathbb{F}^n$

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lemma If

$$f_i(a_j) = \begin{cases} \neq 0 & \text{if } i=j \\ 0 & \text{if } i>j \end{cases}$$

then the f_i are lin. indep.