PROBLEM SESSION

2024-05-03

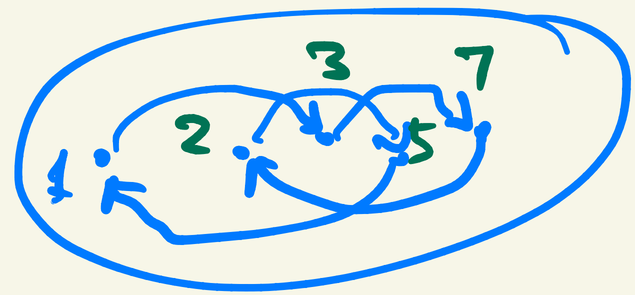
(1

10.24

$c(\sigma, 1)$ $\quad \sigma \in S_n$

length of cycle through 1

[n]



$$P(c(\sigma,1)=k)=\frac{1}{n}$$

bijective proof

					4	6	8
3	7	2	5	1	8	4	6

10.27 $E(\#\text{cycles of } \sigma)$

(S_n, unif)

$X_k = \#\ k\text{-cycles in } \sigma$

$$\boxed{Y = \sum_{k=1}^{n} X_k}$$ Y: # cycles

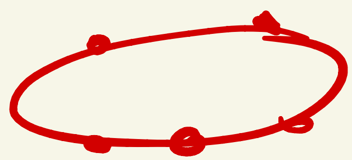
$$X_k = \frac{1}{k} \sum_{i=1}^{n} Y_{ik}$$

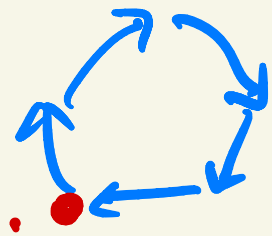
Y_{ik} indicates that $c(\sigma, i) = k$

length of cycle

$$E(Y_{ik}) = P(c(\sigma, i) = k) = \frac{1}{n}$$

$$\boxed{E(X_k) = \frac{1}{k} \underbrace{\sum_{i=1}^{n} \underbrace{E(Y_{ik})}_{1/n}}_{1} = \frac{1}{k}}$$





i

$$E(Y) = \sum_k E(X_k) = \sum_{k=1}^{n} \frac{1}{k} = H_n \sim \ln n$$

harmonic sum $\qquad \int^n \frac{dx}{x}$

LHS ... RHS

10.85 quadruples in p.p.

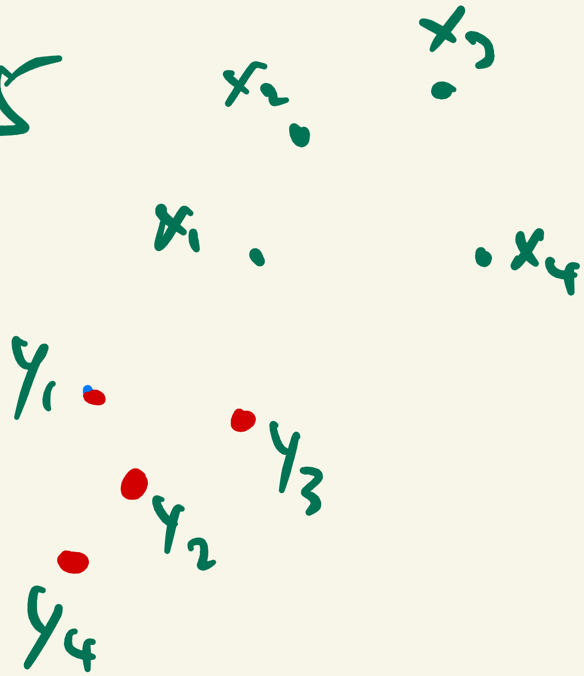
8.52 $\forall$ chain $\leq s \implies m \leq \sum s$ largest binomial coeffi

10.96 small set of gen's in STS

10.88 $|\mathrm{Aut}(\mathrm{Fano})| = 168$

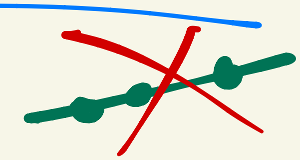
8.31 # chains $< 4 \cdot n^n$

10.85



in $PG(2, \mathbb{F})$

Aut is transitive on ordered quadruples in general position



$\exists \sigma \in Aut(PG\ldots)$ s.t. $x_i^\sigma = y_i$, $i \in [4]$

WLOG

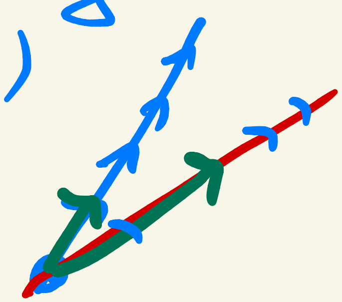
$x_1 = (1:0:0)$

$x_2 = (0:1:0)$

$x_3 = (0:0:1)$

$x_4 = (1:1:1)$

lemma $A \in \mathbb{F}^{3\times3}$, nonsingular ($\exists A^{-1}$) induces an aut. of PG



Lemma

$$A = [\underline{a}_1\ \underline{a}_2\ \underline{a}_3]$$

$$A\underline{e}_i = \underline{a}_i$$

$$e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} \quad e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix} \quad e_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Let $A_1 = [y_1\ y_2\ y_3]$

$$A_1 e_i = y_i$$

$e_1 \to y_1$

of

$e_2 \to y_2$

$e_3 \to y_3$

Need A_2: $[A_2 e_i = \lambda_i e_i]$

$$A_2 f := y_4 \qquad f = e_1 + e_2 + e_3$$

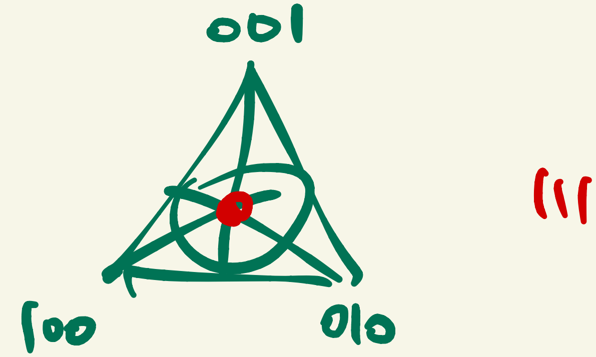
$$\underbrace{\overset{e_1}{e_1},\ \overset{e_2}{e_2},\ \overset{e_3}{e_3},\ \overset{e_4}{y_4}}_{\text{gen. pos.}} = \begin{bmatrix}\beta_1\\\beta_2\\\beta_3\end{bmatrix}$$

gen. pos. $\Longrightarrow (\forall i)(\beta_i \neq 0)$ ✓

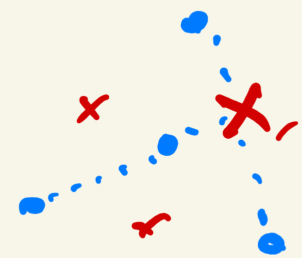
$$\Longrightarrow A_2 = \begin{bmatrix}\lambda_1 & & 0\\ & \lambda_2 & \\ 0 & & \lambda_3\end{bmatrix}$$

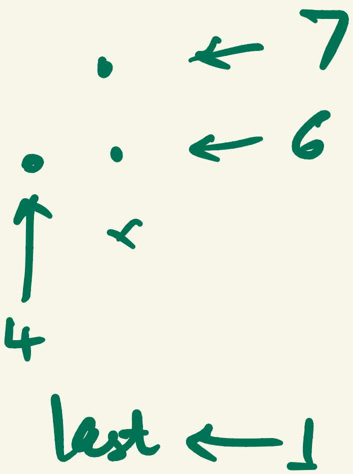
$$A_2 \begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{bmatrix} \qquad \lambda_i := \beta_i$$

$|\mathrm{Aut}(\mathrm{Fano})|$

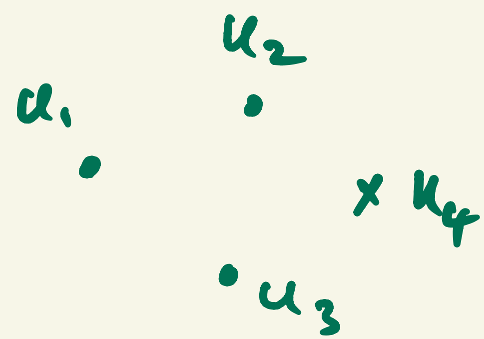


$=$ # quadruples in general position





$\# \; 7\cdot\underline{6}\cdot 4\cdot 1 = 168$



Lemma $\forall$ triangle $\exists$ fourth pt $\underbrace{}_{\text{gen pos}}$

$\exists$ small set of gen's of STS

$a_1 :\quad 1$

$a_2 :\quad 3$

$a_3 :\quad \geq 7$

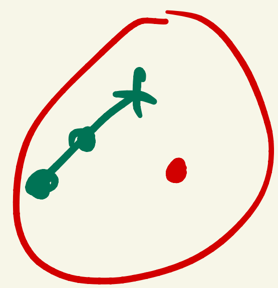
$a_4 \quad \geq 15$

$\vdots$

$$k \downarrow 2k+1$$

STS

$$a_k \geq 2 \cdot a_{k-1} + 1$$



$b_1 = 2$

$b_2 \geq 4$

$\vdots$

$b_k \geq 2^k$

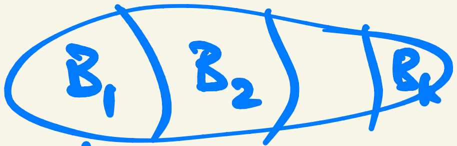
$n = a_k \geq 2^k - 1$

$b_{k-1} = a_{k-1} + 1$

$\underline{\underline{b_k}} = a_k + 1 \geq 2 \cdot a_{k-1} + 2 = 2(a_{k-1}+1) = 2\,\underline{\underline{b_{k-1}}}$

$n+1 \geq 2^k \quad \boxed{k \leq \log_2(n+1)}$

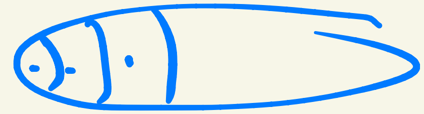
\# chains



reduced chains $\longleftrightarrow$ ordered partition

$\emptyset, \Omega$ not in the chain

\# max chains $n!$



$\#\text{chains} = 4 \times \#\text{reduced chains}$

$B_1 \quad B_1 \cup B_2 \quad \ldots \quad B_1 \cup \ldots \cup B_{k-1}$

$\{\text{ordered partitions}\} \xrightarrow{\text{inj.}} [n]^{[n]}$

$(B_1 \ldots B_k) \longmapsto f : [n] \to [n]$ where $f(x) = i$ if $x \in B_i$

8.52 s-antichain: $\forall$ chain $\leq s$

<u>Claim</u> $\mathcal{F}$: ... $\Longrightarrow |\mathcal{F}| \leq \sum s$ largest binomial coeff's

Pf

Case $s=1$ BLYM: $\sum \frac{1}{\binom{n}{|A_i|}} \le 1$

$X \le 1$

$\therefore E(X) \le 1$

Pf: σ random lin order; X: # A_i prefix of σ

General case: $\ast \sum \frac{1}{\binom{n}{|A_i|}} \le s$

$X \le s$

$\therefore E(X) \le s$

Deduce from this: triv: $m \le s \cdot \binom{n}{\frac{n}{2}}$

$m \le \sum s$ largest binom coeff's

$A_i \prec A_j \Rightarrow \binom{n}{|A_i|} \ge \binom{n}{|A_j|}$

$A_1 \dots A_{\binom{n}{n/2}}$

$G \leq S_n$ permutation group of degree n

$\sigma \in G$

$E(\#\text{fixed points of } \sigma) = \#\,\underline{\text{orbits}}$ of G

Pf if G is transitive

X_i : i is fixed

$Y = \sum X_i$ $\quad E(Y) = \sum E(X_i)$ ⊛

$E(X_i) = \Pr(i^\sigma = i) = \frac{1}{n}$

$E(X_i) = \frac{1}{|i^G|}$

'length of orbit of i

each orb contributes 1 to the sum ⊛

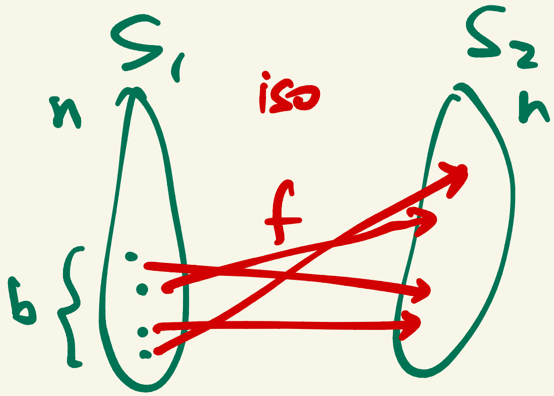
$G_i \leftrightarrow G_{i \to j}$

stab. $\quad i^\tau = j$

$G_{i \to j} = G_i \cdot \tau$

10.92 $|\mathrm{Aut}(S,\circ)| \le n^b$

b: size of smallest set of gen's



maps $[b] \to [n]$ is n^b

$$f(a \circ b) = f(a) \circ f(b)$$

6.65 Σ binomials = Fib

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} = F_{n+1}$$

B(n): # (0,1) strings of length n w/o consecutive 1s

$F_{n+2} = B(n) = \sum_k \boxtimes$

1 0 0 0 1 0 1 0 0 1 0 1

↑ ↑ ↑ ↑ ↑

n−k zeros

n−k+1 slots for the 1s $\boxtimes$ $\binom{n-k+1}{k}$ ways