Linear Programming LP

optimize linear objective function
under linear inequality constraints

STANDARD FORM

$$\underline{x} \geq \underline{0}$$
$$A\underline{x} \leq \underline{b}$$
$$\max \leftarrow \underline{c}^{tr}\underline{x}$$

PRIMAL

$A \in \mathbb{R}^{k \times n}$

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$

$b = \begin{bmatrix} b_1 \\ \vdots \\ b_k \end{bmatrix} \in \mathbb{R}^k$

$c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$

LP

STANDARD FORM

$$\left.\begin{array}{c} \underline{x} \geq \underline{0} \\ A\underline{x} \leq \underline{b} \\ \hline \max \leftarrow \underline{c}^{tr}\underline{x} \end{array}\right\} \text{PRIMAL}$$

$A \in \mathbb{R}^{k\times n}$

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$c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$

$$\left.\begin{array}{c} \underline{y} \geq \underline{0} \\ A^{tr}\underline{y} \geq \underline{c} \\ \hline \min \leftarrow b^{tr}\cdot\underline{y} \end{array}\right\} \text{DUAL}$$

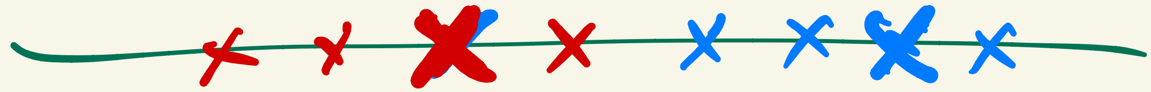
$y \in \mathbb{R}^k$

Weak duality

$$\underset{PR}{\max} \leq \underset{DL}{\min}$$

DUALITY THM

$\max = \min$ $\Longleftarrow$ $\left\{\begin{array}{l} \text{if each is feasible} \\ \rightarrow \exists \max, \min \end{array}\right.$



Weak duality means

for $\forall$ sol $\underline{x}$ of primal
and $\forall$ sol $\underline{y}$ of dual

$$c^{tr} x \le b^{tr} y$$

$Ax \le b \quad x \ge 0$

$A^{tr} y \ge c$

$y^{tr} A \ge c^{T}$

$(AB)^{tr} = B^{tr} A^{tr}$

Pf

$$c^{T} x \le (y^{T} A) x = y^{T} A x = y^{T} (Ax) \le y^{tr} b = (y^{tr} b)^{tr} = b^{tr} y$$

$k \quad k \times n \quad n \quad k \quad k$

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ν^* fractional matching number

τ^* " covering "

$$\tau^* = \min\left\{\underbrace{\sum_{i=1}^{n} x_i} \;\middle|\; (\forall j)\left(\sum_{i\in E_j} x_i \ge 1\right), (\forall i)(x_i \ge 0)\right\}$$

$\underline{x} \ge \underline{0}$ $\quad x \in \mathbb{R}^n$

$M\underline{x} \ge \underline{1}_m \in \mathbb{R}^m$

M: incidence matrix

$m \times n$

#edges $\quad$ # vxs

$\min \underbrace{\underline{1}_n^{tr} \cdot \underline{x}}$

$$\nu^* = \max\left\{\sum_{j=1}^{m} y_j \;\middle|\; (\forall i)\left(\sum_{j\in E_i} y_j \le 1\right), (\forall j)(y_j \ge 0)\right\}$$

E_j

$\underline{y} \ge 0$

$M^{tr}\underline{y} \le \underline{1}$ DUAL

$\max \underline{1}_m^{tr} \cdot \underline{y}$

Str. duality $\nu \le \nu^* = \tau^* \le \tau$

$$\underbrace{\tau^* \le \tau}_{\text{trivial}} \le \tau^* \cdot \underbrace{\;?\;}_{\text{INTEGRALITY GAP}}$$

LÁSZLÓ LOVÁSZ

$$\tau \le \tau^*(1+\ln \deg_{max})$$

Greedy cover algorithm

- pick vertex of max degree
  - remove, along with all incident edges
- repeat

Size of greedy cover : t $\qquad t \ge \tau$

$$t \le \tau^*(1+\ln \deg_{max})$$

ν_k : max k-matching : $\forall$ vertex in at most $\underbrace{\text{selected edges}}_{\text{multiset}}$

$\tilde{\nu}_k$: max $\underbrace{\text{simple}}_{\text{set}}$ k-matching (not multi)

$\boxed{\text{DO}}$! $\qquad \tilde{\nu}_k \leq \nu_k$

$\boxed{\text{DO}}$! $\qquad \nu \leq \frac{\nu_k}{k} \qquad\qquad k \cdot \nu \leq \nu_k$

$\boxed{\text{DO}}$! $\qquad \frac{\nu_k}{k} \leq \nu^*$

$d = \deg_{max}$

THEOREM 2.

$$t \leq \frac{\tilde{\nu}_1}{1\cdot 2} + \frac{\tilde{\nu}_2}{2\cdot 3} + \cdots + \frac{\tilde{\nu}_{d-1}}{(d-1)\cdot d} + \frac{\tilde{\nu}_d}{d}$$

- $\tilde{\nu}_d = m = |\mathcal{E}|$

COROLLARY previous thm

Pf of THM 2. t_k : # times $\deg_{max} = k$

$$t = t_d + t_{d-1} + \cdots + t_1$$

after $t_d + \cdots + t_{i+1}$ steps: $\mathcal{H}_i$ residual hypergr

$\deg_{max}(\mathcal{H}_i) \leq i$ $\quad |\mathcal{E}_i| = \tilde{\nu}_i(\mathcal{H}_i) \leq \tilde{\nu}_i$

LHS $\leq$ RHS

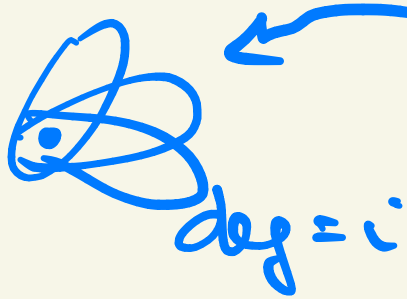
Pf of THM2. t_k : # times $\deg_{max} = k$

$$t = t_d + t_{d-1} + \cdots + t_1$$

after $t_d + \cdots + t_{i+1}$ steps: $\mathcal{H}_i$ residual hypergraph

$$\deg_{max}(\mathcal{H}_i) \le i \qquad |\mathcal{E}_i| = \tilde{\nu}_i(\mathcal{H}_i) \le \tilde{\nu}_i$$

$$|\mathcal{E}_i| = i \cdot t_i + (i-1) \cdot t_{i-1} + \cdots + t_1 \qquad \le \tilde{\nu}_i$$



NTS $\quad t \le \frac{\tilde{\nu}_1}{1\cdot 2} + \frac{\tilde{\nu}_2}{2\cdot 3} + \dots + \frac{\tilde{\nu}_{d-1}}{(d-1)\cdot d} + \frac{\tilde{\nu}_d}{d}$.

WE KNOW $\left\{ \quad i\cdot t_i + (i-1)\cdot t_{i-1} + \dots + t_1 \le \tilde{\nu}_i \right.$ $\quad \times \frac{1}{i(i+1)}$

$i = 1 \dots d-1$

$\times \frac{1}{d}$

$i = d$

NTS: in this lin comb.

$$(\forall i)(\text{coeff}(t_i) = 1)$$

$$\text{coeff}(t_i) = i\Big(\underbrace{\frac{1}{i(i+1)}}_{\frac{1}{i} - \frac{1}{i+1}} + \frac{1}{(i+1)(i+2)} + \dots + \frac{1}{(d-1)d} + \frac{1}{d}\Big)$$

$$\frac{1}{i} - \frac{1}{i+1} + \frac{1}{i+1} - \frac{1}{i+2} + \dots + \frac{1}{d-1} - \frac{1}{d} + \frac{1}{d}$$

telescoping sum

$$= i\cdot\frac{1}{i} = \underline{\underline{1}}$$

THM2 $t \le \frac{\tilde{\nu}_1}{1\cdot 2} + \frac{\tilde{\nu}_2}{2\cdot 3} + \dots + \frac{\tilde{\nu}_{d-1}}{(d-1)\cdot d} + \frac{\tilde{\nu}_d}{d}$

deg max

THM1 $t \le \tau^* \cdot (1 + \ln d)$

actually $t \le \tau^* \cdot H_d$

$H_d = 1 + \frac{1}{2} + \dots + \frac{1}{d} < 1 + \ln d$

harmonic number

$\int \frac{dx}{x}$

$\tilde{\nu}_k \le \nu_k$

$\frac{\nu_k}{k} \le \nu^*$

THM2 $\Longrightarrow$

$t \le \nu^* \Big(\underbrace{\tfrac{1}{2} + \tfrac{1}{3} + \dots + \tfrac{1}{d} + 1}_{H_d} \Big)$

✓

$m(r)$: min # edges of a non-2-colorable
r-unif hypergraph

$m(r) > 2^{r-1}$ was thm

THM (ERDŐS) $m(r) = O(r^2 \cdot 2^r)$

pf probabilistic method

SEMI-CONSTRUCTIVE PF:
follows from greedy cover