HONORS COMBINATORICS

2024-05-09

Birthday paradox

~ 1

$\sim$ even

$- \ \varepsilon$

?

1000

$\ln n$

$\sqrt{n}$

$\frac{n}{100}$

$\sim n$

$f : [k] \to [n]$ $n = 365$ $k = 30$

$\Pr(f \text{ injective})$ $f(x) = f(y) \Rightarrow x = y$

<u>thm</u> If $k = o(\sqrt{n})$ $\quad \frac{k}{\sqrt{n}} \to 0$

then $\Pr(\text{inj}) \to 1$

If $k = \omega(n)$ $\quad \frac{k}{\sqrt{n}} \to \infty$

then $\Pr(\text{inj}) \to 0$

$$\Omega = [n]^{[k]} = \{f : [k] \to [n]\}$$

$$p = \Pr(\text{inj}) = \frac{n(n-1)\cdots(n-k+1)}{n^k} = \frac{1}{1}\cdot\left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)$$

$$p = \Pr(\text{inj}) = \frac{n(n-1)\cdots(n-k+1)}{n^k} = \frac{1}{1}\cdot\left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)$$

FACT $(\forall x \in \mathbb{R})(1+x \le e^x)$

$$p < e^{-\sum_{1}^{k-1} \frac{i}{n}} = e^{-\frac{1}{n}\cdot\binom{k}{2}}$$

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if $k^2 = \omega(n)$ then exponent $\to -\infty$

upper bound $\to 0$

$$k = o(\sqrt{n})$$

FACT

$$0 \le a, b \le 1 \implies (1-a)(1-b) \ge 1-(a+b)$$

$$0 \le a_i \le 1 \implies \prod (1-a_i) \ge 1 - \sum a_i$$

$$p = \prod_{i=1}^{k-1} \left(1 - \frac{i}{n}\right) > 1 - \sum_{i=1}^{k-1} \frac{i}{n} = 1 - \frac{\binom{k}{2}}{n}$$

if $k^2 = o(n)$ then this lower bound $\to 1$

$\boxed{\text{HW}}$ If $k = \Theta(\sqrt{n})$ then $p = \Theta(1)$

$1 - p = \Theta(1)$

i.e. $0 < c_1 \le p \le 1 - c_2 < 1$

LOVÁSZ integrality gap

$\mathcal{H}$

$$\tau^* \leq \underset{\substack{| \\ \text{covering } \#}}{\tau} < \tau^*(1 + \ln \deg_{max})$$

$m(r) =$ min # edges of non-2-colorable r-unif hypp

$$m(r) > 2^{r-1}$$

THM (ERDŐS) $m(r) = O(r^2 \cdot 2^r)$

Pf prob. method

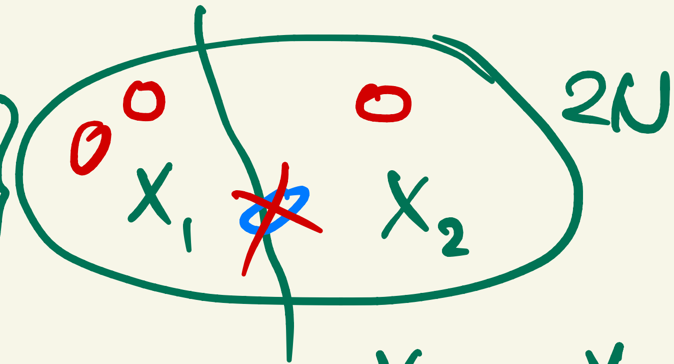
2^{nd} Pf : via LOVÁSZ gap

Pf via Lovász gap

$N \approx r^2$

$V = \binom{2N}{r}$

$\mathcal{E} = \{E(X_1,X_2) \mid X_1 \sqcup X_2 = [2N]\}$



$\mathcal{H} = (V, \mathcal{E})$

$X_1 \sqcup X_2 = [2N]$

$E(X_1,X_2) = \binom{X_1}{r} \sqcup \binom{X_2}{r} \subset V$

Let C be a cover of $\mathcal{H}$

$C \subseteq V = \binom{2N}{r}$

Take $\mathcal{K} = ([2N], C)$

<u>Claim</u> $\mathcal{K}$ is not 2-colorable

Pf of Claim:

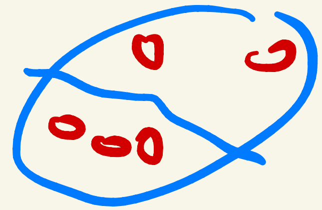
$\{X_1, X_2\}$ not a 2-coloring

C cover $\Rightarrow \exists T$

$C \cap E(X_1, X_2) \ni T$

but T is mono-chr. under 2-coloring (X_1, X_2)

Claim $\underset{\tau(\mathcal{H})}{\overset{\#\text{edges}}{m}} = O(r^2 \cdot 2^r)$ when $N = r^2$



$$|E(X_1, X_2)| = \binom{X_1}{r} + \binom{X_2}{r} \geq 2 \cdot \binom{N}{r}$$

Let $w_x := \dfrac{1}{2 \cdot \binom{N}{r}} \implies \sum_{x \in E} w_x \geq 1$

So $\{w_x \mid x \in V\}$ is a fractional cover of $\mathcal{H}$

$$\therefore \quad \tau^* \leq \frac{|V|}{2\binom{N}{r}} = \frac{\binom{2N}{r}}{2\binom{N}{r}}$$

$$\tau^* \leq \frac{|V|}{2\binom{N}{r}} = \frac{\binom{2N}{r}}{2\binom{N}{r}}$$

$$q := \frac{\binom{2N}{r}}{\binom{N}{r}} = \frac{2N\cdot(2N-1)\cdot\ldots\cdot(2N-r+1)}{N\cdot(N-1)\cdots(N-r+1)} > 2^r$$

NEED upper bd $O(2^r)$

$N = r^2$

$$q < \left(\frac{2N-r+1}{N-r+1}\right)^r = \left(2+\frac{r-1}{N-r+1}\right)^r =$$

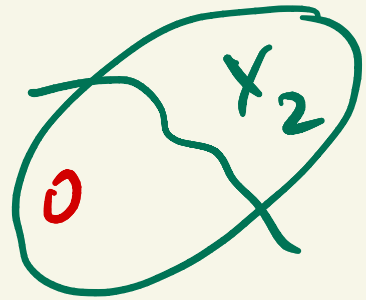
$$2^r\left(1+\frac{r-1}{2(N-r+1)}\right)^r < 2^r \cdot e^{\frac{r(r-1)}{2(N-r+1)}} < 2^r \cdot \sqrt{e}$$

$1+x < e^x$

$$\tau^* \le \frac{9}{2} < 2^r \cdot \sqrt{e}/2$$

$$\tau \le \tau^* (1 + \ln deg_{max})$$

$$deg \le 2^{2N-r}$$



$\mathcal{H}$

$$deg_{max} = 2^{2N-r} < e^{2N-r}$$

$$\ln deg_{max} < 2N-r$$

$$\tau \le 2^r \cdot \underbrace{\sqrt{e} \cdot (2N-r+1)}_{O(r^2)}$$

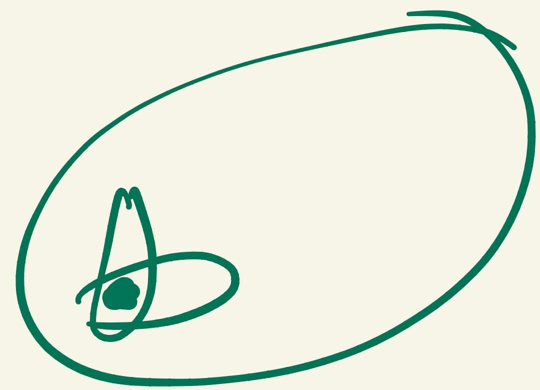
$\mathcal{H}$ is an intersecting hypergraph if

$$(\forall i,j)(E_i \cap E_j \neq \emptyset)$$

ERDŐS-KO-RADO 1936-1964

PAUL ERDŐS
CHAO KO
RICHARD RADO

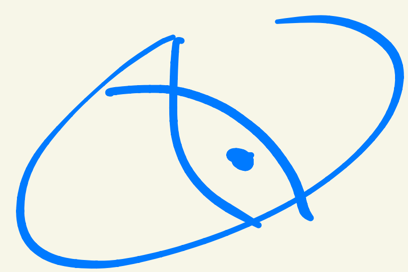
max # edges of r-unif intersecting hyp. with n vertices



$$\max = \binom{n-1}{r-1}$$

$\boxed{\text{for } n \geq 2r}$

if $n < 2r$ then $\binom{n}{r}$



Pf by GYULA KATONA

variation on Lubell's permutation method

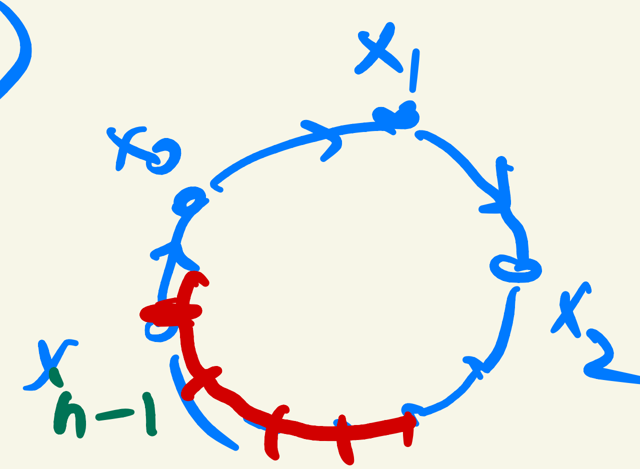
$$\binom{n-1}{r-1} = \frac{r}{n}\binom{n}{r}$$

Take cycle $C = (x_0, x_1, \dots x_{n-1})$ distinct, cyclic order of $[n]$

$= (x_{n-1}, x_0, x_1, \dots x_{r-2})$

$x_i \in [n]$

$E \subseteq [n]$ is compatible with C if it is an arc



$\boxed{\text{HW}}$ $n > 2r$

LEMMA $\# E \in \mathcal{E}$ compatible w C is $\leq r$