PROBLEM SESSION | 2024-05-10 | 1

13.27 Counting $f:[n]\to[k]$ surjections

$$k^n - k\cdot(k-1)^n + \binom{k}{2}(k-2)^n + -$$

$$= \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$$

$i \in [k]$

$A_i : i \notin \mathrm{range}(f)$

$S_j \neq$ # of fctns $|\mathrm{range}| = k-j$

$$\Pr(f \text{ surjective}) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left(1 - \frac{j}{n}\right)^n$$

closed-form for fixed k

13.38 $\mathbb{F}^{\leq d}[x_1 \dots x_n] = \{ f \in \mathbb{F}[x_1, \dots x_n] \mid \deg f \leq d \}$

① subspace: lin comb does not increase degree

$\{\deg f = d\}$
$0 \notin$ —"—
$f + (-f) \notin$
$0 \cdot f \notin$

② $\mathbb{F}^{\leq d}[\underline{x}] = \operatorname{span}\left\{ x_1^{k_1} \cdots x_n^{k_n} \;\middle|\; \begin{array}{l} k_i \geq 0 \\ \underline{\sum k_i \leq d} \end{array} \right\}$

stars + bars sol.

$i = 1 \dots n$ $\quad \underbrace{\times\times\times}_{k_1} \mid \underbrace{\times\times\times\times}_{k_2} \mid\mid \underbrace{\times \cdot \times}_{k_3}$

$\# \times\text{'s} \leq d$
$\# \mid\text{'s} = n-1$

$i = 0 \dots n$ $\quad \sum_{i=0}^{n} k_i = d$

$\# \times\text{'s} = d$
$\# \mid\text{'s} = n$

$\dim = \binom{n+d}{d}$

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$f_1 \dots f_m \in \mathbb{F}^\Omega$

$a_1 \dots a_m \in \Omega$

$f_i(a_j) \in \mathbb{F}$

$\Omega \to \mathbb{F}$

$|\Omega| = n$

$\dim \mathbb{F}^\Omega = n$

$x_1 \quad \dots \quad x_n \quad \Omega$

$x_i \in \mathbb{F}$

$A = (f_i(a_j))_{m \times m}$

Claim If A nonsingular then the f_i are lin. indep.

Suppose $\sum \alpha_i f_i = \underline{0} \qquad \alpha_i \in \mathbb{F}$

NTS $(\forall i)(\alpha_i = 0)$

$\sum \alpha_i f_i(a_j) = 0$

$\begin{bmatrix} \sum \alpha_i f_i(a_1) \\ \sum \alpha_i f_i(a_2) \\ \vdots \end{bmatrix}$ lin comb of columns of A

$\therefore \ \forall \alpha_i = 0$ b/c A nonsing.

Special cases:

① $$f_i(a_j) = \begin{cases} \neq 0 & i = j \\ 0 & i \neq j \end{cases} \qquad \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}$$

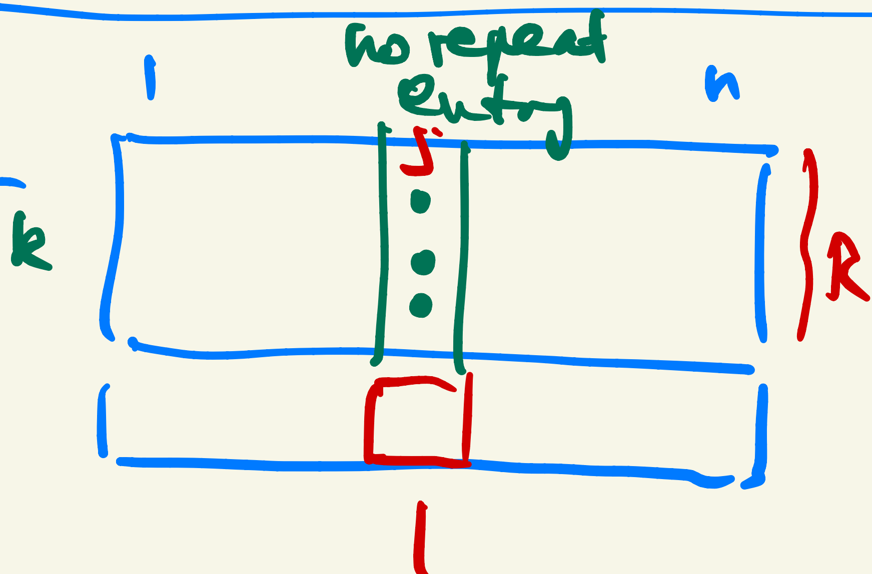
$\Longrightarrow$ the f_i lin indep

② $$f_i(a_j) = \begin{cases} \neq 0 & i = j \\ 0 & i > j \end{cases} \qquad \begin{pmatrix} * & & ? \\ & \ddots & \\ 0 & & * \end{pmatrix}$$

König 1916: $r \geq 1$

r-reg. r-unif hypergraph has SDR

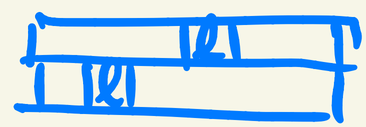
$\forall$ row: perm of $[n]$



$k \times n$

Latin rectangle

R_j: j^{th} column



E_j of permitted entries $|E_j| = n-k$

need SDR of $\{E_j : j \in [n]\}$

$\mathcal{H} = \big([n], \{E_j \mid j \in [n]\}\big)$ Claim: reg. of deg $n-k$

$\ell \in [n]$ belongs to $E_j \Leftrightarrow \ell \notin R_j$

possible $(k+1)^{st}$ rows

$= \#$ SDRs of an $(n-k)$-regular $(n-k)$-unif. hyp

MARSHALL HALL's lemma: r-reg r-unif hyp

$\Rightarrow \#$ SDRs is $\geq r!$

$\therefore \quad L(n) \geq \prod_{r=1}^{n} (r!)$

\# latin squares

12.64 $\ln\left(\prod_{r=1}^{n} r!\right) \sim \frac{1}{2} n^2 \ln n$

$$S = \sum_{r=1}^{n} \ln(r!) < \sum \ln(r^r) < \sum \ln(n^r) =$$

$$= \sum r \ln n = \ln n \sum_{r=1}^{n} r = \binom{n+1}{2} \ln n$$

$$\sim \frac{n^2}{2} \ln n$$

$$\boxed{(*)\ S \gtrsim (1-\varepsilon) \ln n \cdot \frac{n^2}{2}}$$

$$S \lesssim \frac{n^2}{2} \ln n$$

NTS $(\forall \varepsilon > 0)$(for all suff large n)

$$S \gtrsim (1-\varepsilon) \frac{n^2}{2} \ln n$$

$$r! > \left(\frac{r}{e}\right)^r$$

$$S = \sum_{1}^{n} \ln(r!) > \sum_{n^{1-\varepsilon}}^{n} \ln(r!) > \sum_{n^{1-\varepsilon}}^{n} r(\ln r - 1) = \sum_{n^{1-\varepsilon}}^{n} r \ln r - \underbrace{\sum r}_{< \binom{n+1}{2}}$$

$$\sum_{r=n^{1-\varepsilon}}^{n} r \ln r > \sum r \cdot \ln n^{1-\varepsilon} = (1-\varepsilon) \ln n \sum_{n^{1-\varepsilon}}^{n} r = (1-\varepsilon) \ln n \left[\binom{n+1}{2} - \binom{n^{1-\varepsilon}}{2}\right]$$

$(*)$

Stolz-Cesáro Thm:
discrete L'Hôpital's

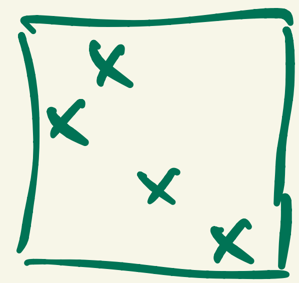
12.112 stochastic matrix $A=(a_{ij})$

$a_{ij} \geq 0$, $\sum_{j=1}^{n} a_{ij} = 1$ ∀ row is a probab. distrib.

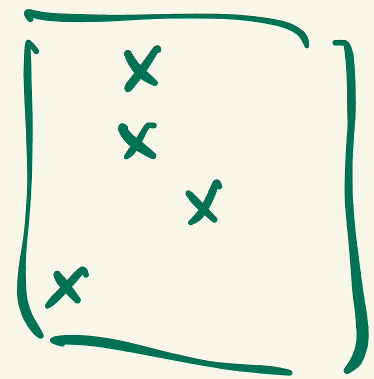
Claim per $A \leq 1$

more generally, if $A \geq 0$ (∀entry ≥ 0)

then per $A \leq \prod$ row sums



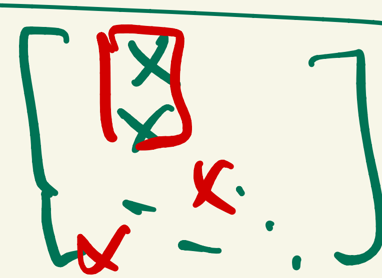
per $A \le \prod$ row sums



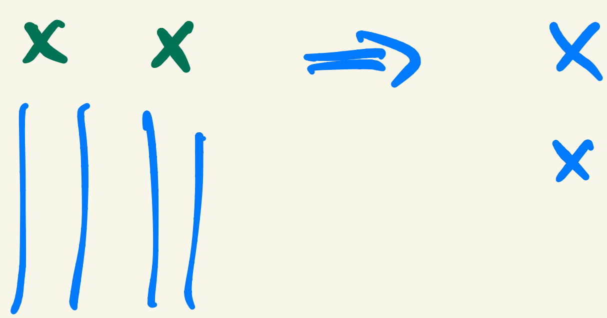
$$\text{RHS} = \sum_{f \in [n]^{[n]}} a_{i f(i)}$$

$$\text{LHS} = \sum_{f \in S_n} a_{i f(i)} \quad \le \text{RHS} \checkmark$$

Lemma If per $A = 1$ then



$\therefore$ $\forall$ column ≤ 1 entry not zero



If $\forall$ other row avoids these two col's $\Rightarrow$ by PHP $\exists$ two nonzeros in same col.

THM <u>Permanent Inequality</u> <u>Egorychev Falikman</u>

$J = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \vdots & & & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$

If A is doubly stoch $\to \operatorname{per} A \geq \operatorname{per}\left(\frac{1}{n} J\right) = \frac{n!}{n^n}$

$> e^{-n}$

Tóger Bang

12.126 r-reg, r-unif hyp. has $> \left(\frac{r}{e}\right)^n$ SDRs

Pf M incidence matrix

r-unif $\Longrightarrow \forall$ row sum $= r$

r-reg $\Longrightarrow \forall$ col. sum $= r$

$\downarrow$

$\frac{1}{r} M$ doubly stoch. $e^{-n} < \operatorname{per}\left(\frac{1}{r} M\right) = \frac{1}{r^n} \operatorname{per} M$

$\# \text{SDR's} \overset{\text{DO}}{=} \operatorname{per} M > r^n \cdot e^{-n} = \left(\frac{r}{e}\right)^n$ ✓

$$\therefore \quad L(n) > \prod_{r=1}^{n} \left(\frac{r}{e}\right)^{n}$$

$$\ln L(n) > \sum_{r=1}^{n} n \cdot (\ln r - 1) = \overbrace{\Big(n \underbrace{\sum \ln r}_{\substack{\ln n! \\ \sim n \ln n}}\Big)}^{\sim n^2 \ln n} - n^2 \sim n^2 \ln n$$

$$\therefore \quad \ln L(n) \gtrsim \underline{n^2 \cdot \ln n}$$

$$\ln L(n) < \ln\left(n^{n^2}\right) = \underline{n^2 \ln n}$$

<u>12.129</u>

$$\therefore \quad \ln L(n) \sim n^2 \ln n$$

10.115 BON strongly neg. correl. events

$A_1 \dots A_m$ events s.t. $\Pr(A_i) = \frac{1}{2}$

$\Pr(A_i \cap A_j) < \frac{1}{5}$

Claim $m \le 6$

Pf X_i : indicator of A_i $\quad X_i : \Omega \to \mathbb{R}$

$\operatorname{Var} Y = E((\dots)^2)$

$Y := \sum X_i$

$$\underline{0 \le} \operatorname{Var} Y = \sum_i \sum_j \operatorname{Cov}(X_i, X_j)$$

$$= \underbrace{\sum \operatorname{Var} X_i}_{\frac{n}{4}} + \underbrace{\sum_{i \ne j} \operatorname{Cov}(X_i, X_j)}_{-\frac{n(n-1)}{20}} \ge 0$$

$5 - (n-1) \ge 0$

$6 \ge n$ ✓

$\operatorname{Cov}(R, S) = E(RS) - E(R)E(S)$

r.v.'s

$\operatorname{Var}(R) = \operatorname{Cov}(R, R)$

B: Bernoulli trial w prob p of success

indicator var. $\longrightarrow B^2 = B$

$1^2 = 1$

$0^2 = 0$

$E(B) = p$

$\mathrm{Var}(B) = E(B^2) - E(B)^2 = p - p^2 = \underline{p(1-p)}$

$p = \frac{1}{2}$ $\quad$ $\mathrm{Var} = \frac{1}{4}$

$$\mathrm{Cov}(X_i, X_j) = \underbrace{E(X_i X_j)}_{\substack{P(A_i \cap A_j) \\ \leq \frac{1}{5}}} - \underbrace{E(X_i)}_{\frac{1}{2}} \cdot \underbrace{E(X_j)}_{\frac{1}{2}} \leq \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}$$

X_i : ind A_i

X_j : ind A_j

$X_i X_j$: ind $A_i \cap A_j$

$$\alpha \cdot \chi \geq n \quad \checkmark$$

10.78

If $\mathcal{H}$ is vertex-trans. then

$$\alpha \cdot \chi \leq n(1+\ln n)$$

MARIO SZEGEDY

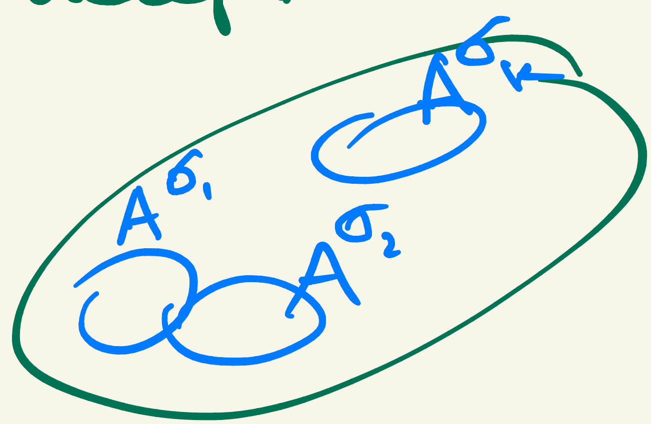
chrom. # = min k s.t. $V = \bigcup_{i=1}^{k}$ indep sets

Pf probabilistic method

Let $A \subseteq V$ $|A| = \alpha$ A indep.

Pick random $\sigma_i \in \mathrm{Aut}(\mathcal{H})$

A^{σ_i} indep.



$x \in V$

$$\Pr(x \in A^{\sigma}) = \Pr(x^{\sigma^{-1}} \in A) = \frac{\alpha}{n}$$

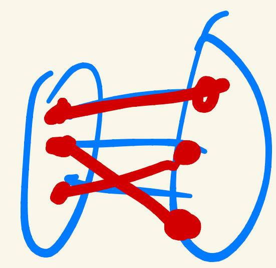
12.37 König Duality $\Longrightarrow$ Marriage Thm

for bipartite graphs $\tau = \nu$

covering #   matching #

If $(\forall T \subseteq A)$
$(|N(T)| \geq |T|)$
$\Longrightarrow \exists$ matching of size $|A|$

i.e. $\nu = |A|$



Assume $\nu < |A|$

Then $\tau < |A|$ (Kőnig)

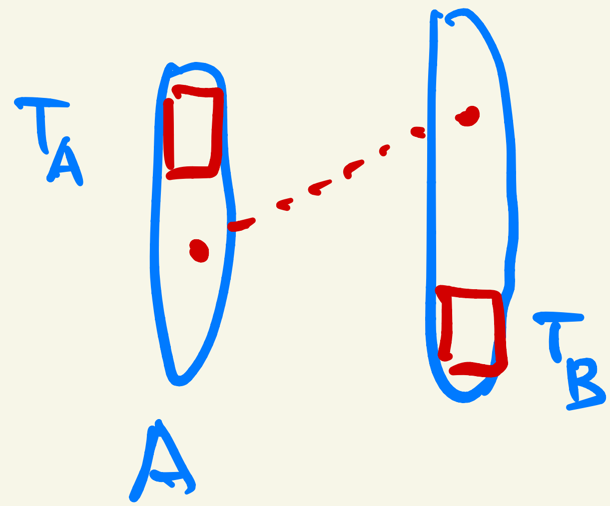
Choose $|T| = \tau$

NTS: Hall violation

candidate violator

$$|A \setminus T_A| =: t$$

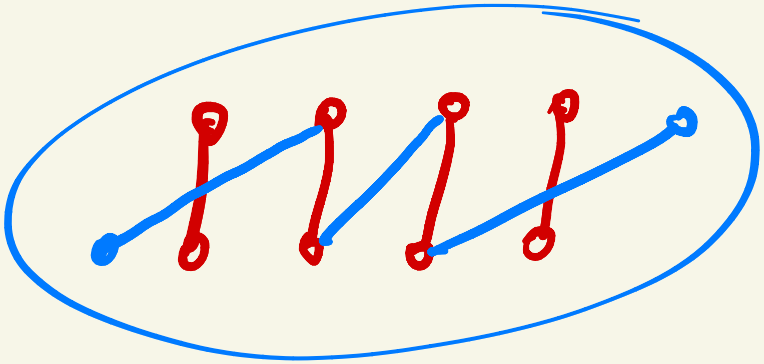
$$N(A \setminus T_A) \subseteq T_B$$



$$\underline{|N(A \setminus T_A)|} \le |T_B| = |T| - |T_A|$$

$$= \tau - |T_A| < \underline{|A| - |T_A|}$$

$T = T_A \cup T_B$



augmenting path
can be found by BFS
in a directed graph

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$G = \mathrm{Aut}(\mathcal{H})$

$$\Pr_{\sigma \in G}(x \in A^{\sigma}) = \frac{\alpha}{n}$$

$1 + x < e^{x}$

$$\Pr(x \notin A^{\sigma}) = 1 - \frac{\alpha}{n}$$

$$\Pr(x \notin A^{\sigma_1} \cup \dots \cup A^{\sigma_k}) = \left(1 - \frac{\alpha}{n}\right)^{k} < e^{-\frac{\alpha k}{n}}$$

$$\Pr(\exists x \in V)(\text{—} \text{''} \text{—}) < n \cdot e^{-\frac{\alpha k}{n}}$$

So if $n e^{-\frac{\alpha k}{n}} \le 1$ then $\chi \le k$

i.e. $n \le e^{\alpha k / n}$

$$\ln n \le \frac{\alpha k}{n}$$

$$\frac{n}{\alpha} \cdot \ln n \le k$$

$$\therefore \chi \le \left\lceil \frac{n}{\alpha} \ln n \right\rceil$$

$$< \frac{n}{\alpha} \cdot (1 + \ln n)$$

✓