HONORS COMBINATORICS

2024-05-14

r-uniform hypergraph $m \le 2^{r-1} \Longrightarrow$ 2-colorable

Conjecture ERDŐS: $\exists c>0$ s.t. if $\mathcal{H}$ is an r-uniform hypergraph and every edge intersects at most $c \cdot 2^r$ edges

then $\mathcal{H}$ is 2-colorable)

This will follow from the

LOVÁSZ LOCAL LEMMA

LLL

If $A_1, \dots, A_m$ events

$x_1, \dots, x_m \in (0,1)$

s.t. $(\forall i)\left(\Pr(A_i) \le x_i \prod_{j \in N(i)} (1 - x_j)\right)$

neighbor

then

$$P\left(\cap \bar{A}_i\right) \ge \prod_{i=1}^{m} (1 - x_i) > 0$$

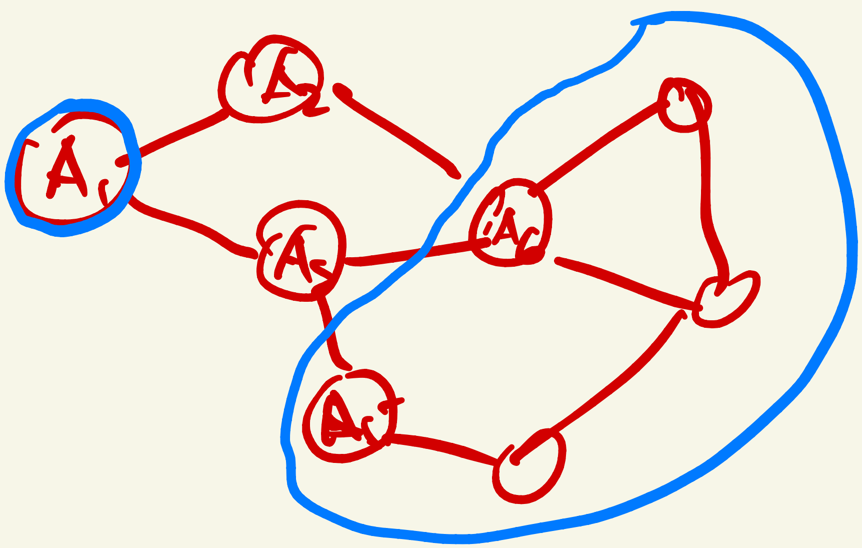
Graph

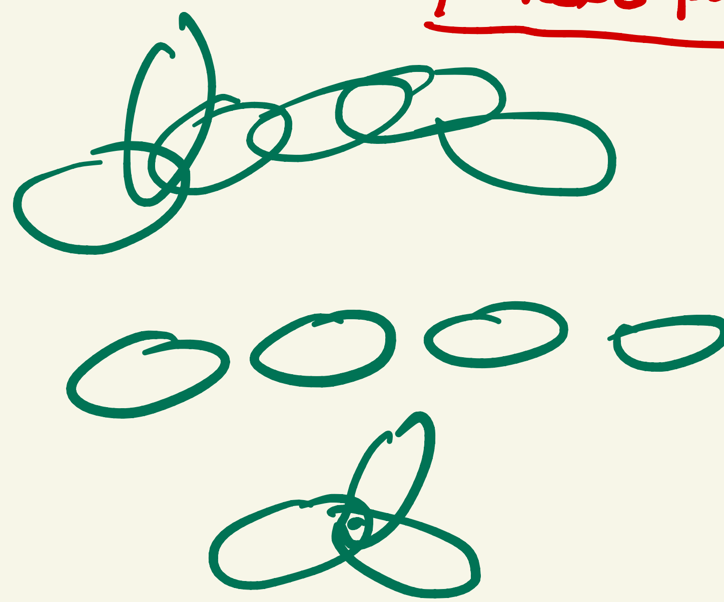
$G = ([m], E)$

s.t. $\forall i$

A_i is <u>indep.</u> of $\{A_j \mid j \nsim i\}$

see DEF next page





DEF. Independence of sets of events

$$\underbrace{A_1, \ldots A_m}_{\text{these are indep. of}}, \underbrace{B_1, \ldots, B_k}_{\text{these}}$$

these are indep. of these

if their indicator variables are

DEF Independence of sets of random variables

rv's $\underbrace{X_1 \ldots X_m}$ $\underbrace{Y_1 \ldots Y_k}$

these are indep. of these if

$\forall x_1 \ldots x_m \quad y_1 \ldots y_k \in \mathbb{R}$

$$P(X_1 = x_1 \wedge \cdots \wedge X_m = x_m \wedge Y_1 = y_1 \wedge \cdots \wedge Y_k = y_k)$$

$$= P(\text{— " —}) \cdot P(\text{— " —})$$

COROLLARY to LLL

If $(\forall i)\left(\Pr(A_i) \le \frac{1}{e(d+1)}\right)$ then $P(\cap \bar{A}_i) > 0$. $\qquad d = \deg_{\max}(G)$

Pf $\quad x_i := \frac{1}{d+1} \qquad \Pr(A_i) \overset{?}{\le} \frac{1}{d+1}\cdot\left(1-\frac{1}{d+1}\right)^d > \frac{1}{e(d+1)} \ge P(A_i)$

$$\left(1-\frac{1}{d+1}\right)^d = \left(\frac{d}{d+1}\right)^d = \frac{1}{\left(1+\frac{1}{d}\right)^d} \qquad < e$$

COROLLARY E. conj.

r-unif., # edges intersecting an edge $< \frac{2^r}{2e} - 1 \Rightarrow$ 2-colorable

A_i : ith edge monochr. $\quad \Pr(A_i) = \frac{1}{2^{r-1}} < \frac{1}{e(d+1)}$

ERDŐS-LOVÁSZ 1975 $\quad\|\quad$ Algorithmic version ~ 2010 MOSER-TARDOS

Claim $\forall S \subseteq [m] \setminus \{i\}$

$$\Pr\Big(A_i \,\Big|\, \underbrace{\bigcap_{j\in S} \bar{A}_j}_{\Pr>0}\Big) \le x_i$$

events C_i

$C_1 C_2 \cdots C_k := C_1 \cap C_2 \cap \ldots \cap C_k$

Claim $\Rightarrow$ LLL :

$$P\left(\bigcap \bar{A}_i\right) = \Pr(\bar{A}_1)\cdot \Pr(\bar{A}_2 \mid \bar{A}_1) \cdots \Pr(\bar{A}_m \mid \bar{A}_1 \cdots \bar{A}_{m-1})$$

$$\ge (1-x_1)\cdot(1-x_2)\ldots(1-x_m)$$

L1 $\Pr(B_1 B_2 \ldots B_k) = P(B_1)\cdot P(B_2 \mid B_1)\cdot P(B_3 \mid B_1 B_2) \ldots P(B_k \mid B_1 \ldots B_{k-1})$

L1 $\Pr(A \mid B \cap C) = \dfrac{\Pr(A \cap B \mid C)}{\Pr(B \mid C)}$ assuming $\Pr(B \cap C) > 0$

Claim $\forall S \subseteq [m]\setminus\{i\}$

$$\Pr\Big(A_i \;\Big|\; \bigcap_{j\in S} \bar{A}_j\Big) \le x_i$$

$S_1 = N(i)\cap S$

$S_2 = S \setminus S_1$

Proof: induction on $|S|$

$$\Pr\Big(A_i \;\Big|\; \bigcap_{j\in S}\bar{A}_j\Big) \overset{L2}{=} \frac{\Pr\big(A_i \cap \bigcap_{S_1}\bar{A}_j \;\big|\; \bigcap_{S_2}\bar{A}_j\big)}{\Pr\big(\bigcap_{S_1}\bar{A}_j \;\big|\; \bigcap_{S_2}\bar{A}_j\big)} = \frac{\text{Num}}{\text{Denom}}$$

$$\text{Num} \le \Pr\Big(A_i \;\Big|\; \bigcap_{S_2}\bar{A}_j\Big) = P(A_i) \le x_i \prod_{j\in N(i)}(1-x_j) \quad *$$

assn in LLL

$$\text{Denom} \overset{L1}{=} \Pr(\bar{A}_{j_1} \mid S_2)\cdot\Pr(\bar{A}_{j_2} \mid \bar{A}_{j_1}\cap S_2)\cdot\Pr(\bar{A}_{j_3} \mid \bar{A}_{j_1}\bar{A}_{j_2}\cap S_2)\cdots$$

IH $\longrightarrow$ $\ge (1-x_{j_1})\cdot(1-x_j)\cdot(1-x_{j_3})\cdots \;\; \ge \prod_{S_1}(1-x_j) \quad *$

$S_1 = \{j_1, \ldots, j_k\}$

DO

L1 $\Pr(B_1B_2\ldots B_k) = P(B_1)\cdot P(B_2 \mid B_1)\cdot P(B_3 \mid B_1B_2)\cdots P(B_k \mid B_1\ldots B_{k-1})$

L2 $\Pr(A \mid B\cap C) = \dfrac{\Pr(A\cap B \mid C)}{\Pr(B \mid C)}$ assuming $\Pr(B\cap C) > 0$