HONORS COMBINATORICS

2024-05-16

1

Vandermonde matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & & & & \alpha_n \\ \alpha_1^2 & \alpha_2^2 & & & & \alpha_n^2 \\ \vdots & & & & & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \cdots & & & \alpha_n^{n-1} \end{pmatrix} =: V_n(\alpha_1 \ldots \alpha_n)$$

$\boxed{\text{DO}}$ nonsingular prove w/o determinant

If $\alpha_i = \alpha_j \to$ singular

$(\forall i \neq j)(\alpha_i \neq \alpha_j) \implies$ nonsingular

$\deg f = n$
$\implies$ has $\leq n$ roots

DEF points in $\mathbb{R}^n$ in <u>general position</u> if $\forall n$ of them are <u>lin. independent</u>

$$f : \mathbb{R} \longrightarrow \mathbb{R}^n$$

$$f(t) = \begin{pmatrix} f_0(t) \\ \vdots \\ f_{n-1}(t) \end{pmatrix} := \begin{pmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^{n-1} \end{pmatrix}$$

moment curve

points in general position

i.e. $t_1, \dots, t_n$ distinct $\in \mathbb{R}$

$$\Longrightarrow f(t_1), \dots, f(t_n) \in \mathbb{R}^n$$

b/c $V_n(t_1 \dots t_n)$ is nonsing.

α, χ, τ, ν

~1960 TIBOR GALLAI → LOVÁSZ's mentor

$\mathcal{H} = (V, \mathcal{E})$ is τ-critical if

$\forall\, \mathcal{E}' \subsetneq \mathcal{E},\ \mathcal{H}' = (V, \mathcal{E}')$

$$\tau(\mathcal{H}') < \tau(\mathcal{H})$$

Obs $\forall$ hypergraph $\mathcal{H}$ contains a subhypergraph $\mathcal{H}'$

removing edges only

s.t.

$\tau(\mathcal{H}') = \tau(\mathcal{H})$

but $\mathcal{H}'$ is τ-critical

r-uniform τ-critical hypergraph

with $\tau(\mathcal{H}) = s+1$

$\Longrightarrow$ $m \le f(r,s)$

edges

$\mathcal{E} = \{A_1 \dots A_m\}$

$\mathcal{E}_i = \mathcal{E} \setminus \{A_i\}$ $\quad (\exists B_i)(\underline{|B_i| = s} \wedge$

$A_j \cap B_i \neq \emptyset$ if $i \neq j$

$A_i \cap B_i = \emptyset$ b/c $\tau(\mathcal{H}) = s+1$

$A_1 \ \dots \ A_m$ $\qquad |A_i| = r$

$B_1 \ \dots \ B_m$ $\qquad |B_i| = s$

$$A_i \cap B_j = \begin{cases} \emptyset & \text{if } i = j \\ \neq \emptyset & \text{if } i \neq j \end{cases}$$

Gallai-Bollobás conditions

Take case $r = s$

we can get $\rightarrow \binom{2r}{r}$

take $n = 2r$

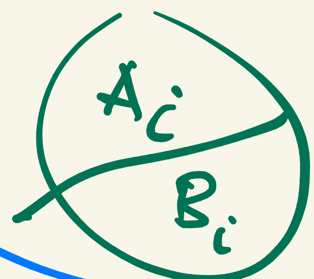
$A_i \in \binom{[2r]}{r}$

$B_i = \overline{A}_i$

general r, s

we can get

$$\binom{r+s}{r}$$



ERDŐS - HAJNAL · MOON 1964

optimal for $s=2$

BÉLA BOLLOBÁS 1965: $\binom{r+s}{r}$ optimal

14

Thm 1 Under GALCAI conditions $m \leq \binom{r+s}{r}$

Thm 2 If $A_1 \ldots A_m$, $B_1 \ldots B_m$ $\quad A_i \cap B_j = \begin{cases} \emptyset & i=j \\ \neq\emptyset & i\neq j \end{cases}$

then

$$\sum_{i=1}^{m} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1$$

$\boxed{\text{DO}}$ $\Longrightarrow$ BLYM

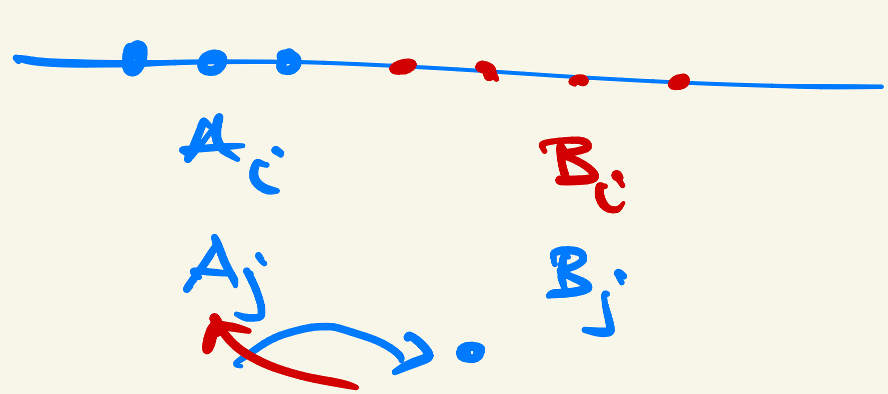
COR uniform case Thm 1

Proof via Lubell's method

σ linear order of vertices

(σ, i) compatible pair if $\max A_i < \min B_i$

$X(\sigma) = \#\{i \mid (\sigma, i) \text{ compat}\}$



① $X \leq 1$

② $\sigma \in S_n$ random

X random var.

$X = \sum_{i=1}^{m} Y_i$ $\quad$ Y_i indicates (σ, i) compat.

$$1 \geq E(X) = \sum E(Y_i) = \sum_i P(\sigma, i \text{ comp}) = \sum_i \frac{1}{\binom{|A_i| + |B_i|}{|A_i|}}$$

threshold version of Gallai condition

$$|A_i \cap B_j| = \begin{cases} \leq t & \text{if } i=j \\ > t & \text{if } i \neq j \end{cases}$$

FÜREDI

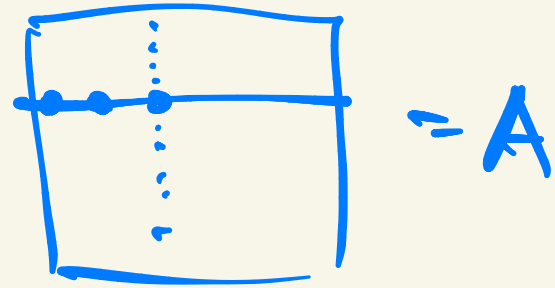
$$m \leq \binom{r+s-2t}{r-t}$$

lin alg method works in skew setting

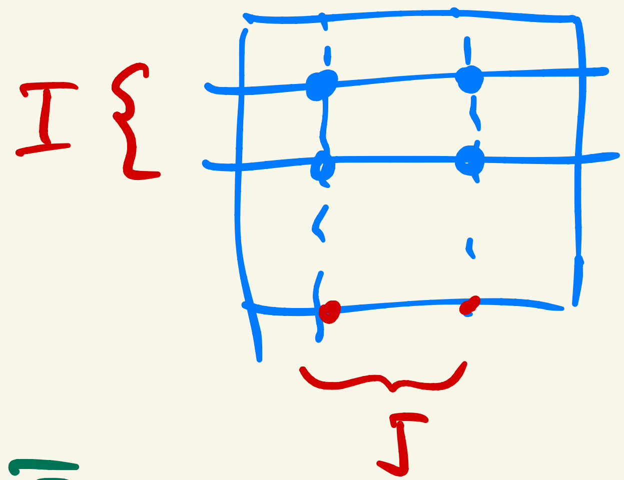
$$A_i \cap B_j = \begin{cases} \emptyset & \text{if } i=j \\ \neq \emptyset & \text{if } i<j \end{cases}$$

Lovász proof of uniform version

Laplace expansion of det



± sum of $\binom{n}{r}$ terms



$I, J \subseteq [n]$, $|I| = r$

$$\det A = \sum_J (-1)^e \det_I A_J \cdot \det_{\bar{I}} A_{\bar{J}}$$

(fix I) $\quad \underline{e = \sum I + \sum J}$

$\underline{x}, \underline{y} \in \mathbb{R}^{\binom{n}{r}}$

$$\underline{x} \cdot \underline{y} = \sum_{J \in \binom{[n]}{r}} (-1)^{\sum I + \sum J} x_J \cdot y_J$$

$V = \{v_1 \dots v_n\} \subseteq \mathbb{R}^{r+s}$ general position

$$\underline{a}_i = \begin{bmatrix} \underline{w}_1 \\ \vdots \\ \underline{w}_r \end{bmatrix} \leftarrow \text{corresp. to elements of } A_i$$

$\underline{w}_j \in V$

$r \times (r+s)$ matrix

$\underline{x}_i = \bigwedge$ rows of A_i

$$C(i,j) = \begin{array}{c} r \\ s \end{array} \begin{bmatrix} A_i \\ \hline B_j \end{bmatrix} \begin{array}{l} \mapsto \underline{x}_i \in \mathbb{R}^{\binom{r+s}{r}} \\ \mapsto \underline{y}_j \in \end{array}$$

$$r+s$$

if $i \neq j$ then $A_i \cap B_j \neq \emptyset$

$\det C(i,j) = 0$

if $i = j$: rows lin indep

$\det C(i,j) \neq 0$

$$\underline{x}_i \cdot \underline{y}_j = \begin{cases} \neq 0 & i = j \\ 0 & i \neq j \quad i<j \end{cases}$$

$\boxed{\text{DO}}$ $\therefore$ the $\underline{x}_i$ are lin. indep $\therefore$ $n \leq \dim = \binom{r+s}{r}$ ✓

11

$$v_i \in \mathbb{R}^{r+1}$$

$$\updownarrow \perp$$

$$A_i$$

$$w_i$$

$$\updownarrow$$

$$B_i$$

$$\underline{x} \in \mathbb{R}^{r+1} \qquad v_i \perp A_i$$

$$f_i(\underline{x}) = \prod_{j \neq i} x \cdot \underline{v}_j$$

$$f_i(w_j) = \begin{cases} \neq 0 & i = j \\ 0 \text{ if } & i \neq j \end{cases}$$

the f_i lin indep

homogeneous poly. of deg r in s var?

$$d_i = \binom{(r+1)+(s-1)}{r} = \binom{r+s}{r}$$