16.51 $\binom{x}{k}$ $x \geq k-1$ convex

$$f(x) = k!\binom{x}{k} = \prod_{i=0}^{k-1}(x-i)$$

$$f'(x) = f(x)\sum_{i=0}^{k-1}\frac{1}{x-i}$$

$x > k-1$

$$f''(x) = f'(x)\left(\sum\frac{1}{x-i}\right) - f(x)\sum\frac{1}{(x-i)^2} =$$

$$f(x)\left(\left(\sum\frac{1}{x-i}\right)^2 - \sum\frac{1}{(x-i)^2}\right)$$

$$\left(\sum a_i\right)^2 - \sum a_i^2 = \sum_{i\neq j} a_i a_j > 0$$

$a_i > 0$

2

Sol #2

Lemma If $f, g > 0$, $f', g' > 0$, $f'', g'' \geq 0$

$$\Longrightarrow f \cdot g > 0, \ (fg)' > 0, \ (fg)'' > 0$$

$$(fg)' = f'g + fg' > 0$$

$$(fg)'' = f''g + 2f'g' + fg'' > 0$$

$(x-i)' = 1 > 0$

$(x-i)'' = 0$

3

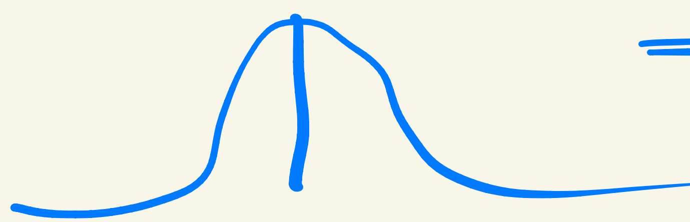
16.45 $\displaystyle \binom{-\frac{1}{2}}{k} = \frac{\prod_{i=0}^{k-1}\left(-\frac{1}{2}-i\right)}{k!} = (-1)^k \frac{\binom{2k}{k}}{4^k}$

$\displaystyle \sim (-1)^k \cdot \frac{1}{\sqrt{\pi k}}$ Stirling's

---

$\displaystyle \sum_{i=0}^{2k} \binom{2k}{i} = 4^k$

$\displaystyle \binom{2k}{k} > \frac{4^k}{2k+1}$



$\sigma \in S_n$ random

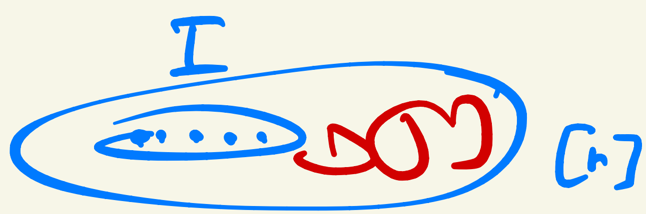
$\Pr(\sigma \text{ is fixed-point-free})$

INCLUSION-EXCLUSION

$A_i : i^\sigma = i$

$$\Pr\left(\bigcap \overline{A_i}\right) = S_0 - S_1 + S_2 - \dots$$

$$S_k = \sum_{I \in \binom{[n]}{k}} \Pr\left(\bigcap_{i \in I} A_i\right) = \binom{n}{k} \frac{(n-k)!}{n!}$$

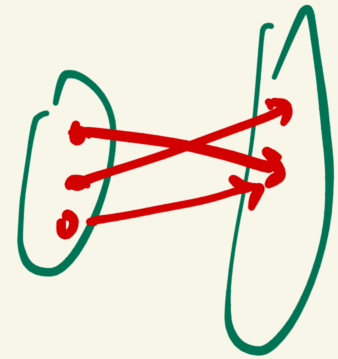


$$= \frac{n!}{k!(n-k)!} \frac{(n-k)!}{n!} = \underline{\underline{\frac{1}{k!}}}$$

$$\Pr = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \dots + (-1)^n \frac{1}{n!} \longrightarrow \frac{1}{e}$$

16.32 Birthday paradox

$$p(n,k) = P(f:[k]\to[n] \text{ injective})$$



$$= \prod_{i=0}^{k-1}\left(1-\frac{i}{n}\right)$$

(*) Claim: $\varepsilon_1 < p(n,k_n) < 1-\varepsilon_2$

$\varepsilon_1, \varepsilon_2 > 0$ constants

(1) upper bd: $1+x < e^x$ $(x \neq 0)$

$$p(n,k) < \prod_{i=0}^{k-1} e^{-\frac{i}{n}} = e^{-\sum \frac{i}{n}} = e^{-\frac{1}{n}\binom{k}{2}}$$

$$< e^{-\frac{1}{n}\binom{c_1\sqrt{n}}{2}} \quad \sim -c_1^2/2$$

$$\to e^{-c_1^2/2} < 1-\varepsilon_2$$

if $k_n = \omega(\sqrt{n})$

$\Longrightarrow p(n,k_n) \to 0$

if $k_n = o(\sqrt{n})$

$\Longrightarrow p(n,k_n) \to 1$

(*) $k_n = \Theta(\sqrt{n})$

$$c_1\sqrt{n} < k_n < c_2\sqrt{n}$$

(2) lower bound

$$p(n,k) = \prod_{i=0}^{k-1}\left(1-\frac{i}{n}\right) > \left(1-\frac{k-1}{n}\right)^k > \left(1-\frac{k}{n}\right)^k \qquad k = c_k\sqrt{n}$$

$$= \left(1-\frac{c_k}{\sqrt{n}}\right)^{c_k\sqrt{n}} = \left[\left(1-\frac{c_k}{\sqrt{n}}\right)^{\frac{\sqrt{n}}{c_k}}\right]^{c_k^2} \sim e^{-c_k^2} >$$

$$\downarrow$$

$$\frac{1}{e}$$

$$e^{-c_2^2} > \varepsilon_1$$

$$\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x = \frac{1}{e}$$

13.68 REINHOLD BAER's THM

$\mathcal{P}_i = (P_i, L_i, I_i)$ $\quad$ $i = 1, 2$

points, lines, incidence relation $I \subseteq P \times L$

DEF dual isomorphism $\mathcal{P}_1 \to \mathcal{P}_2$: isomorphism $\mathcal{P}_1 \to \mathcal{P}_2^{\text{dual}}$

$g : P_1 \to L_2$

$h : L_1 \to P_2$

s.t. (g, h) preserve incidence :

$$p \multimap \ell \iff h(\ell) \multimap g(p)$$

in $\mathcal{P}_1$ $\qquad$ in $\mathcal{P}_2$

DEF $\mathcal{P} = (P, L, I)$ polarity

$g : P \to L$ bijection

s.t. (g, g^{-1}) preserves incidence

$$p \multimap \ell \iff g^{-1}(\ell) \multimap g(p)$$

DEF p is a fixed point of the polarity g if $\quad p \multimap g(p)$

DEF <u>polarity</u> of $\mathcal{P} = (P, L, I)$

bijection $g : P \longrightarrow L$

s.t. $p \mathrel{-\!\!\bullet\!\!-} \ell \iff g^{-1}(\ell) \mathrel{-\!\!\bullet\!\!-} g(p)$

DEF p <u>fixed pt</u> if $p \mathrel{-\!\!\bullet\!\!-} g(p)$

Pf g polarity

$\iff M = M^{tr}$

symmetric

$$M = \begin{array}{c} \\ \ell_1 \\ \vdots \\ \ell_N \end{array} \begin{array}{c} p_1 \quad \cdots \quad p_N \\ \left[\begin{array}{ccc} & & \\ & \boxed{1} & \\ & & \end{array} \right] \end{array}$$

$\ell_i = g(p_i)$

i fixed $\iff$ (i,i) entry $= 1$

fixed-pt-free: diag $= 0$

$N = n^2 + n + 1$

9

Claim If incid. matrix M of p.p. is symm

then $\mathrm{Tr}(M) \neq 0$

For any incid. matrix of $\mathcal{P}$ $\quad MM^T = \begin{pmatrix} n+1 & & 1 \\ & \ddots & \\ 1 & & n+1 \end{pmatrix}$

$= J + nI \qquad J = (\text{all ones})$

---

eigenbasis of J: $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = b_0 \quad \lambda_0 = N = n^2+n+1$

$(\forall i > 0) \quad \lambda_i = 0$

basis: $b_1 \dots b_{N-1}$ of $\{\sum x_i = 0\}$

"zero-weight subspace"

$\cancel{0\,0\,1\,0\,1\,1\,0}$

$\cancel{0\,0\,0\,1\,1\,0\,0}$

$\uparrow$

1

$\therefore b_0 \dots b_{N-1}$ eigenbasis of $\underline{J + nI} \rightarrow \quad \mu_0 = N + n = (n+1)^2$

$\mu_i = n$

$$\mu_0 = N + n = (n+1)^2$$
$$\mu_i = n$$

$$MM^{tr} = M^2$$

eigenv. M : $\rho_0 \cdots \rho_{N-1}$ eigenbasis of M is
" of M^2

$b_0 = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ e.v. of M w e.v. : $\rho_0 = n+1$

$b_i \perp b_0$

$$\rho_i^2 = \mu_i = n$$
$$\rho_i = \pm\sqrt{n}$$

Assume for $\rightarrow\leftarrow$ $\operatorname{Tr} M = 0$

$$n+1 + k\sqrt{n} = 0 \qquad k \in \mathbb{Z}$$
$$n+1 = -k\sqrt{n}$$
$$1 \equiv (n+1)^2 = k^2 \cdot n \equiv 0 \mod n \qquad \rightarrow\leftarrow$$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad A^2 = \underline{\underline{0}}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

"Numbers and Games" book by John Horton Conway

$$\mathcal{P}_1 \subsetneq \mathcal{P}_2$$

subplane

order n

BAER $\Longrightarrow$ order $\leq \sqrt{n}$

$\exists$ inf many cases where a subplane of order $\sqrt{n}$ exists

---

$PG(\sqrt{q},2) \subset PG(q,2)$

$q = p^k$

$\mathbb{F}_q$ field

$GF(q)$

order q

k even:

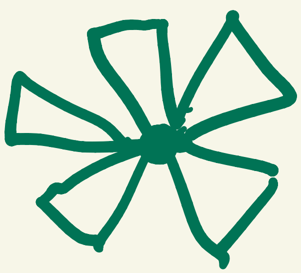
$k = 2l$

$q' = p^l$

13.84 FRIENDSHIP THM

RÉNYI - SÓS - TURÁN

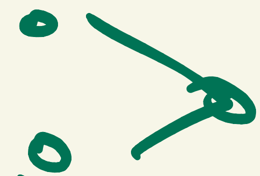
G graph : $(\forall x \neq y \in V)(\exists!\, z)(x \sim z \sim y)$



flower graphs

PF

fp-free polarity

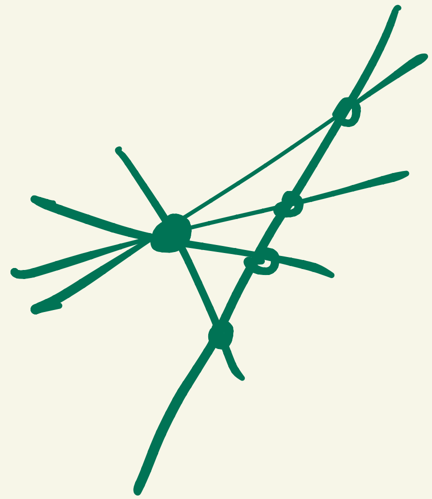


$v \leftrightarrow N(v) = \{w \in V \mid w \sim v\}$

$\{N(v) \mid v \in V\}$ — this is the set of lines in a possibly degenerate p.p.

we get a

$\therefore$ degenerate p.p.



$\Rightarrow$ flower graph

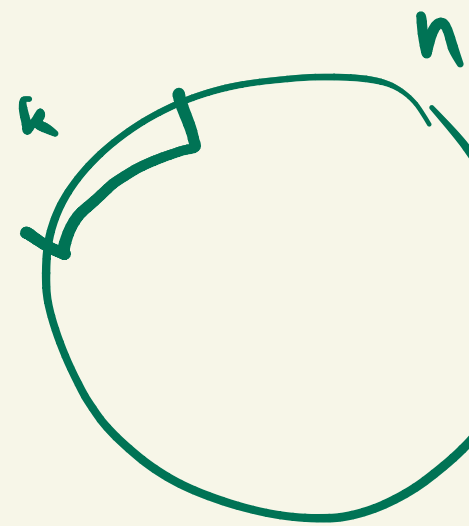
✓

16,61

KATONA'S LEMMA

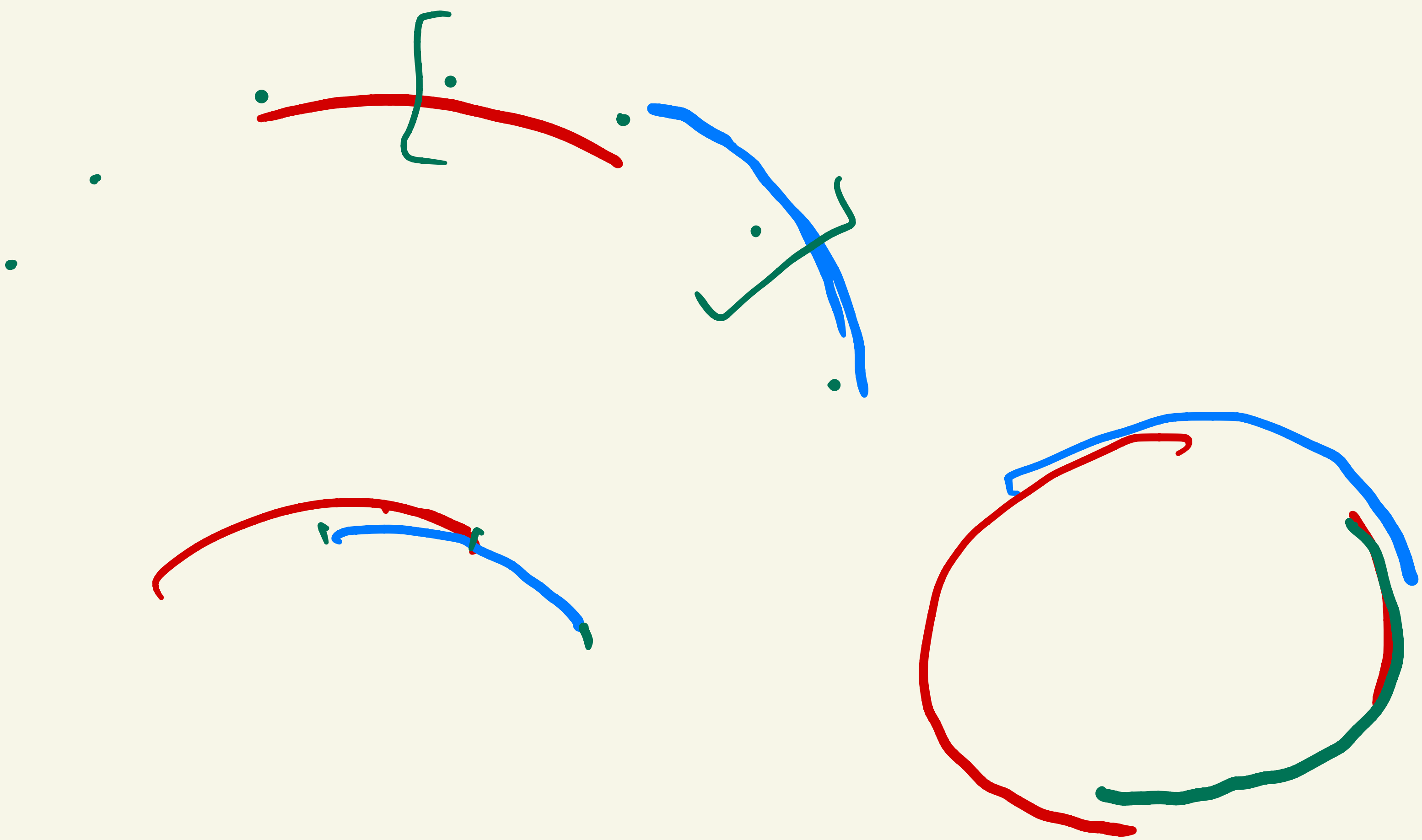
$C_1 \ldots C_m$ are pairwise intersecting arcs of length k

$\Longrightarrow$ $m \leq k$



$2k \leq n$

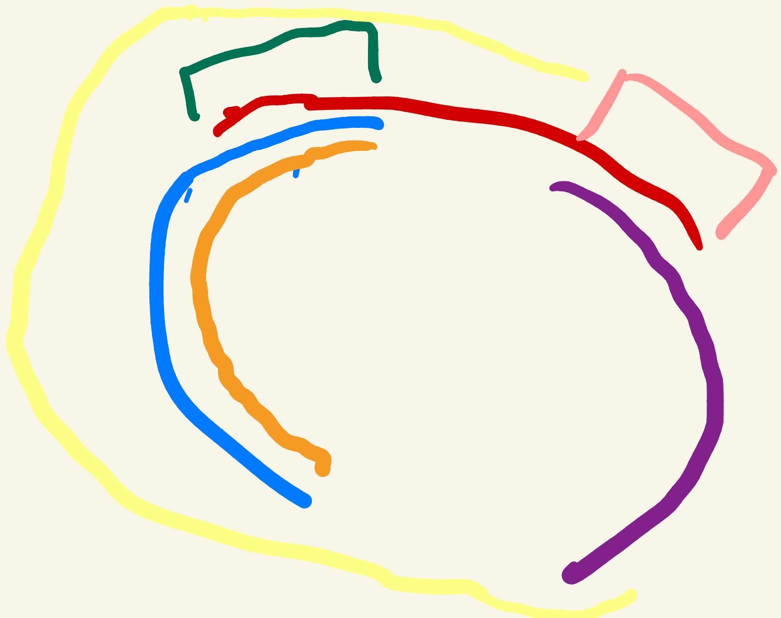
$k=3$



sets $\{C_1 \cap C_i \mid C_i \text{ left}\} \cup \{C_1 \cap \overline{C_j} \mid C_j \text{ right}\}$

all distinct

are starting from left end of C_1



16.48 Integer preserving polynomial $f(x)$:

$$f : \mathbb{Z} \to \mathbb{Z}$$

$$\sum c_k \binom{x}{k} \qquad c_k \in \mathbb{Z}$$

Claim If f is int. pres then $\exists (c_k)$ s.t. $f = \sum c_k \binom{x}{k}$

$\boxed{\text{CH}}$ f is congruence preserving:

(1) f is integer preserving

(2) $(\forall x, y, m \in \mathbb{Z})(x \equiv y \ (m) \Rightarrow f(x) \equiv f(y) \ (m))$

Characterize these polynomials in terms of the coefficients c_k [IMRE RUZSA]