polynomials

$$f(x) = a_0 + a_1 x + \dots + a_n x^n \qquad a_i \in \mathbb{R}$$

DEF

if $a_n \neq 0$ then $\deg(f) = n$

$$\deg(f) = \max\{i \mid a_i \neq 0\}$$

$\deg(0) = ?$

↑ zero polynomial: all coeffs 0

$$\deg(f \cdot g) = \deg f + \deg g$$

$\deg(f+g) = \max(\deg f, \deg g)$

↑ FALSE

	f	g	$f+g$
counterexample:	$\underbrace{x+1}$,	$-x$,	1
deg:	1	1	0

$f(x) = a_0 + a_1 x + \cdots + a_n x^n$

f has deg $= 0 \iff$ all coeff's are zero except $\underline{\underline{a_0}}$

$\underline{\text{i.e.}}$ $a_0 \neq 0$ and $(\forall i \geq 1)(a_i = 0)$

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f is a nonzero constant

$$\deg(f\cdot g) = \deg f + \deg g$$

$$\deg(f+g) \leq \max(\deg f, \deg g)$$

$$k := \deg(0) := \,?$$

$$\deg(\underbrace{f\cdot 0}_{0}) = \underbrace{\deg(f)} + \deg(0)$$

take $f : \deg f = 7$

$$k = k + 7 \Rightarrow k \notin \mathbb{R}$$

$k = \pm\infty$ only candidates

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pick $f \neq 0$

$$\deg(\underbrace{f + (-f)}_{0}) \leq \deg f + \underbrace{\deg(-f)}_{=\deg f} = 2\deg f$$

$\therefore$ only reasonable convention is

$$\boxed{\deg(0) = -\infty}$$

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

$$\mathbb{N}_0 = \{0\} \cup \mathbb{N} = \{0, 1, 2, \dots\}$$

non-negative integers

$$(\forall \text{poly } f)(\deg f \in \mathbb{N}_0 \cup \{-\infty\})$$

$\boxed{\text{Model:}}$

arithmetic with reals

cost of adding/subtracting reals 0

multiplying reals 1

$$\text{cost}(f \cdot g) = (n+1)^2$$

$$\deg f = \deg g = n$$

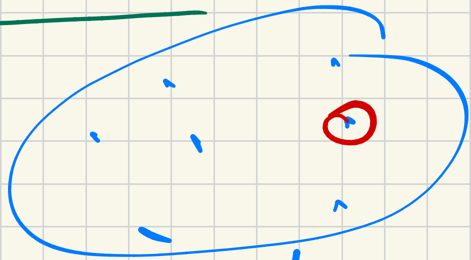
$$(a_0 + a_1 x + \dots + a_n x^n)(b_0 + b_1 x + \dots + b_n x^n)$$

$\cancel{n^2}$ $\quad (n+1)^2 \sim n^2$

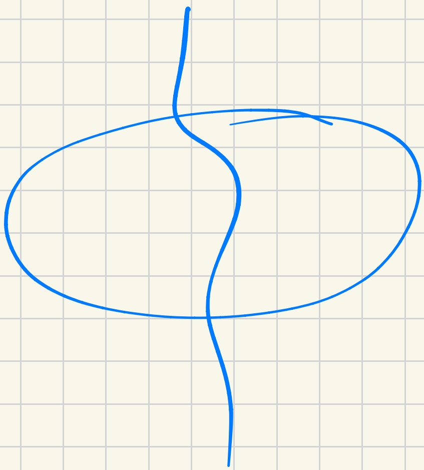
quadratic in n

CAN WE DO BETTER?

identify an object
out of a set of N objects
by Y/N questions



N



k questions

$\downarrow$

2^k answer sequences

$\downarrow$ pigeon hole principle

$$N \le 2^k$$

$$k \ge \log_2 N$$

$$\therefore \quad k \ge \lceil \log_2 N \rceil$$

information theory lower bound

try recursion

$\deg(f) := 2^k - 1$ $\quad$ $n := 2^k$: #coefficients

for convenience: $\deg f = n+1$

$$f = \underbrace{f_1}_{2^{k-1}} + \underbrace{\qquad}_{2^{k-1} = \frac{n}{2}} \cdot x^{\frac{n}{2}} \quad \leftarrow \text{\# coeff's}$$

$$\deg f_1 = 2^{k-1} - 1 = \deg f_2$$

$$f = f_1 + f_2 \cdot x^{\frac{n}{2}}$$

$$g = g_1 + g_2 \cdot x^{\frac{n}{2}}$$

$$f \cdot g = \underline{f_1 g_1} + (\underbrace{f_1 g_2 + f_2 g_1}) x^{\frac{n}{2}} + \underline{f_2 g_2} \cdot x^n$$

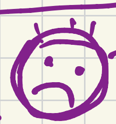
complexity $T(n)$: multipl. two poly's of deg $n-1$

$$\boxed{T(n) \le 4 \cdot T\left(\frac{n}{2}\right)}$$

will be justified later

look for a sol. in the form of $T(n) = n^c$

$$n^c = 4 \cdot \left(\frac{n}{2}\right)^c \qquad 2^c = 4 \qquad c = 2$$



still quadratic

KARATSUBA algorithm:

$$T(n) \leq 3 \cdot T\left(\frac{n}{2}\right)$$

we can do by
3 instances of $\frac{1}{2}$-size problem

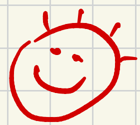
recursively
compute

$$f_1 \cdot g_1 \,,\quad f_2 \cdot g_2 \,,\quad (f_1 - f_2)(g_1 - g_2)$$

$$= \underbrace{f_1 g_1 - (f_1 g_2 + f_2 g_1) + f_2 g_2}$$

$$n^c = 3\left(\frac{n}{2}\right)^c$$

$$2^c = 3 \qquad c = \log_2 3 \sim 1.58$$



"DIVIDE AND CONQUER"