HONORS ALGORITHMS | 2025.01.22

$$\sqrt{n^2+1} - n \sim \frac{1}{2n}$$

$$a^2 - b^2 = (a+b)(a-b)$$

Proof:

$$\sqrt{n^2+1} - n = \frac{1}{\sqrt{n^2+1}+n} \Leftarrow a-b = \frac{a^2-b^2}{a+b}$$

$$\sqrt{n^2+1} \sim n$$

$$n = n$$

$$\boxed{\sqrt{n^2+1} + n \sim 2n}$$

Pf: divide by n

$$\sqrt{1+\frac{1}{n^2}} + 1 \to \left.\begin{matrix}1\\1\end{matrix}\right\} = 2 \checkmark$$

LEMMA If $a_n \sim x_n$ and $a_n, b_n > 0$

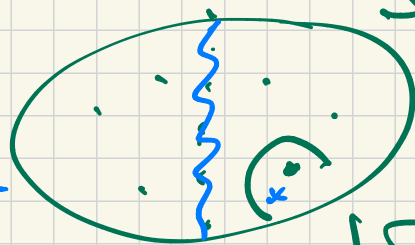
$b_n \sim y_n$

then $a_n + b_n \sim x_n + y_n$ $\boxed{\text{DO}}$

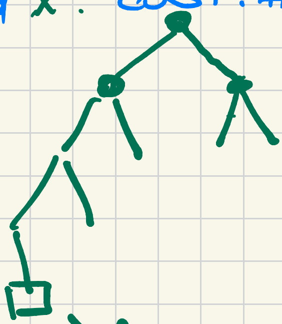
DECISION TREE

Alice and Bob agree on a set Ω of n objects.
Alice selects an object $x \in \Omega$
Bob asks Y/N questions to find x. COST: # questions asked

Ω



$|\Omega| = n$



binary decision tree

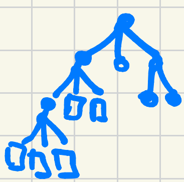
leaves
leaf

$2^{20} = 1{,}048{,}576$

depth $d \implies n \leq 2^d$

$$\boxed{d \geq \log_2 n}$$

Information theory lower bound



ternary decision tree

$$\boxed{d \geq \log_3 n}$$

Suppose each internal node has ≤ 3 children

$\Longrightarrow$ # leaves $\leq 3^d$

$\therefore \quad 3^d \geq n$

$$\boxed{d \geq \log_3 n}$$

$n \to \infty$

SORTING : input : list of reals
output : sorted list
$x_1 \le x_2 \le \dots \le x_n$

COMPARATOR decides: $x \overset{?}{\le} y$
input x, y reals

Cost : # comparisons
bookkeeping : free

WLOG all items are distinct
algorithm produces permutation $\sigma : [n] \to [n]$
bijection

possible outputs $n!$

$$\boxed{\therefore \ \# \text{comparisons} \ge \log_2(n!)}$$

evaluate $\log_2(n!)$ asymptotically

Theorem $\log_2(n!) \sim n\log_2 n$

1st proof: log of Stirling's formula

2nd proof

LEMMA $(\forall n \geq 1)\quad n! \geq \left(\frac{n}{e}\right)^n$

$$\therefore\quad \left(\frac{n}{e}\right)^n \leq n! \leq n^n$$

↖ Lemma ✓

$$\underbrace{n(\log_2 n - \log_2 e) \leq \log_2 n! \leq n\log_2 n}_{\sim}$$ ✓

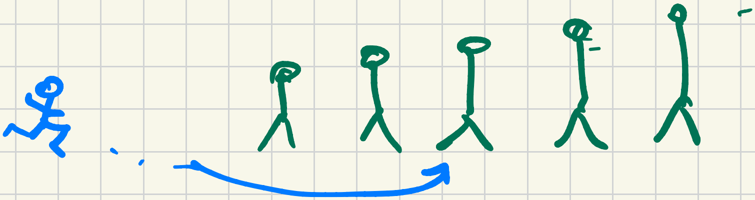
squeeze

$\boxed{\text{CH}}$ $\log_2(n!) - n\log_2 n \sim\ ?$

cn

$c = ?$

INSERTION SORT



binary search to insert next item

complexity $\sim \log_2 1 + \log_2 2 + \dots + \log_2 (n-1)$

$\boxed{\text{DO}}$

$$= \log_2\left((n-1)!\right) \sim (n-1)\log(n-1) \sim n \log n$$

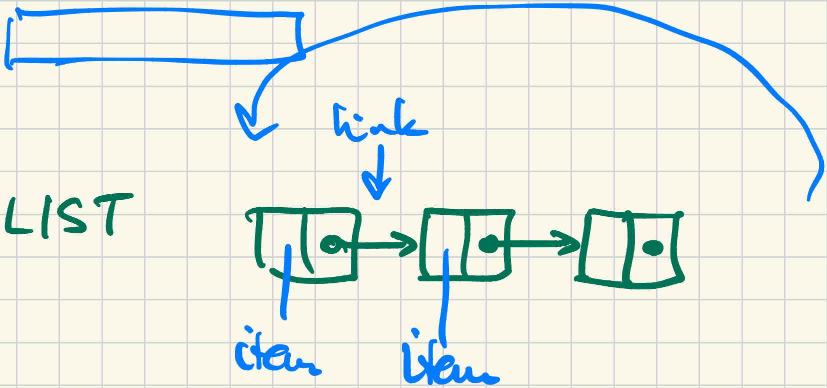
1 2 3 4 5

DATA STRUCTURE

A:

ARRAY

LINKED LIST



INSERT : $O(1)$

FIND : $O(n)$

$$A[1] \le A[2] \le \cdots \le A[n]$$

$$A\left[\left\lfloor \frac{1+n}{2} \right\rfloor\right]$$

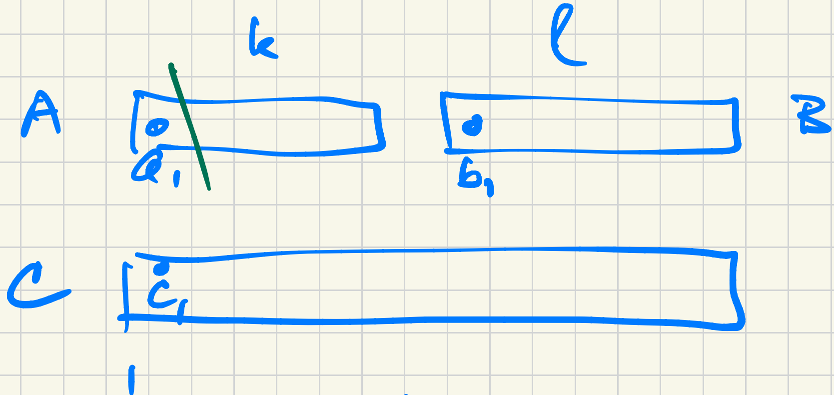
FIND : $O(\log n)$

INSERT : $O(n)$

k ℓ items

MERGE $[A, B]$ A, B sorted arrays

Output: merged, sorted



$c_1 = \min(a_1, b_1)$

$\boxed{\text{DO}}$ Write elegant pseudocode for MERGE

\# comparisons $k+\ell-1$

MERGE-SORT $(A[r \ldots s])$

halving the input $\begin{cases} A_1 = A\left[r, \ldots, \left\lfloor \frac{r+s}{2} \right\rfloor\right] \\ A_2 = A\left[\left\lfloor \frac{r+s}{2} \right\rfloor + 1, \ldots, s\right] \end{cases}$

$MS[A] = \text{MERGE}[MS[A_1], MS[A_2]]$

recursive calls

Cost

$T(n) \leq 2T\left(\frac{n}{2}\right) + n - 1$

Sol'n:

$\sim n \log_2 n$

T : # comparisons

S : — " — + bookkeeping

$$S(n) \leq 2S\left(\frac{n}{2}\right) + O(n)$$

soln $O(n \log n)$

JOHN von NEUMANN