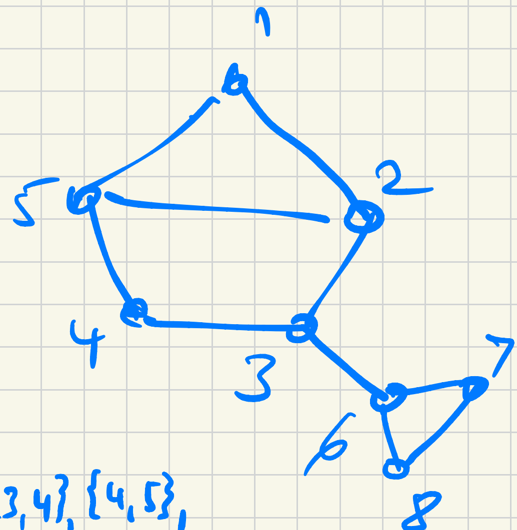
HONORS ALGORITHMS | 2025-01-27

GRAPH THEORY



nodes : vertices
vertex

links : edges

$V = [7]$

$E = \{\{1,2\}, \{2,3\}, \{2,5\}, \{3,4\}, \{4,5\}, \{3,6\}, \{6,7\}, \{7,8\}, \{6,8\}\}$

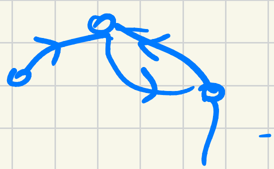
$E = \{12, 23, 25, 34, 45, 36, 67, 78, 68\}$

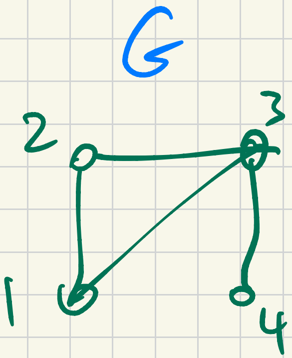
$\{3,6\} = \{6,3\}$

(UNDIRECTED)

(V,E) graph (w/o picture)

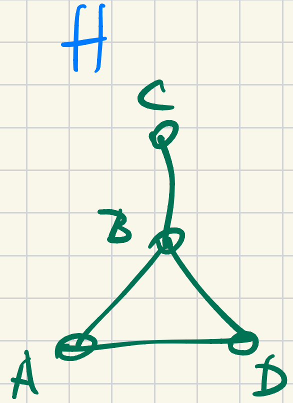
DIGRAPH





$1 \sim 3$

$1 \not\sim 4$



$1 \to D$

$2 \to A$

$3 \to B$

$4 \to C$

If $x, y \in V$ and $xy \in E$ then we say x, y are **adjacent**

$x \sim y$

adjacency relation

$G = (V, E)$ $\quad H = (W, F)$

$f : V \to W$ bijection is an <u>isomorphism</u> of G to H iff preserves **adjacency**

i.e.

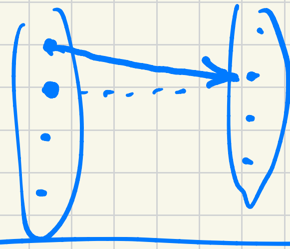
$$(\forall x, y \in V)\left(x \sim_G y \iff f(x) \sim_H f(y)\right)$$

G, H are <u>isomorphic</u> $\quad \exists f : G \to H$ isomorphism

If $|V| = |W| = n$

then there are bijections $V \to W$

$n(n-1) \cdots 3 \cdot 2 \cdot 1 = \underline{\underline{n!}}$

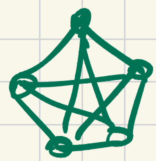


n # vertices

m # edges

$$0 \le m \le \binom{n}{2} = \frac{n(n-1)}{2}$$

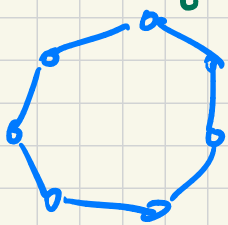
K_5



<u>Complete</u> graph K_n : n vertices

$m = \binom{n}{2}$ edges

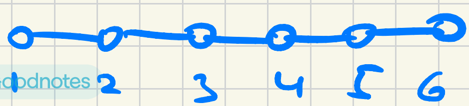
<u>cycle</u> C_n $n \ge 3$ cycle of length n



C_7

<u>path</u> P_n

of length $n-1$



P_6

$x \sim_G y$

Complement of G : $\bar{G}$

$G = (V, E)$ $\qquad$ $\bar{G} = \left(V, \binom{V}{2} \setminus E\right)$ *

set S

$$\binom{S}{k} = \{T \subseteq S \mid |T| = k\}$$

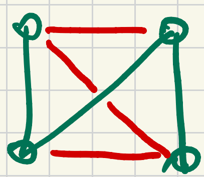
$|S| = r$

$$\left|\binom{S}{k}\right| = \binom{r}{k}$$

$E(G)$ set of edges of G

$E(K_n) = \binom{[n]}{2}$

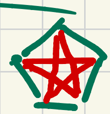
let $V(K_n) := [n]$



complete graph $E = \binom{V}{2}$

* $(\forall x, y \in V)(x \sim_{\bar{G}} y \iff x \not\sim_G y \text{ and } x \neq y)$

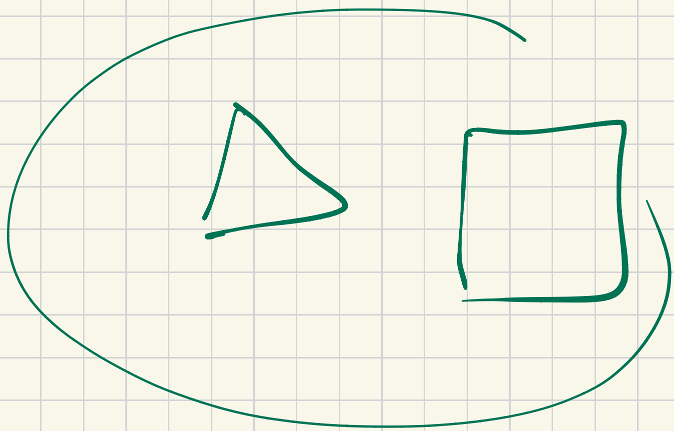
$P_4 \cong \bar{P}_4$ $\qquad$ $C_5 \cong \bar{C}_5$



Thm If G is self-complementary

i.e. $G \cong \overline{G}$ \cong

then $n \equiv 0$ or $1 \mod 4$



- P_4 C_5

Reward:

$\exists$ n-vertex self-compl. graph

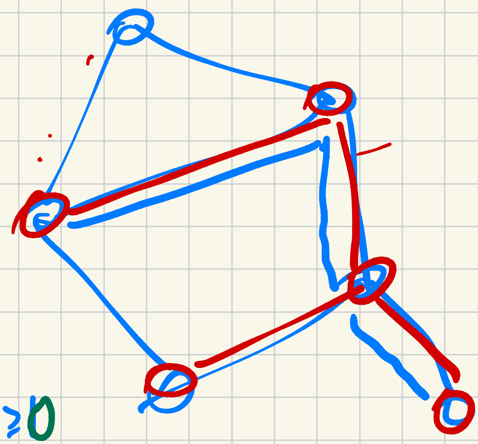
$\iff n \equiv 0$ or $1 \mod 4$

Subgraph: $H = (W, F)$ is a <u>subgraph</u> of $G = (V, E)$

if $W \subseteq V$

and $F \subseteq E$

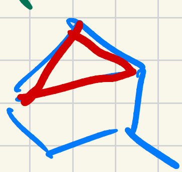
A path in G of length k is a subgraph $\cong P_{k+1}$ $\quad k \geq 0$

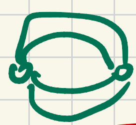


P_4

A cycle of length k in G is a subgraph $\cong C_k$ $\quad k \geq 3$

A <u>clique</u> of size k in a <u>multigraph</u> is a complete subgraph with k vertices





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max clique size <u>NP-hard</u>

P vs NP conjecture

Hamilton cycle in a graph:

cycle that passes through each vertex

i.e. a C_n subgraph

Existence of H-cycle: NP-complete