ALGORITMS PROBLEM SESSION

2025-01-31

p1

7.15 Prove: $n! > \left(\frac{n}{e}\right)^n$ ✓ $n \geq 1$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} > \frac{x^n}{n!}$$

$x > 0$

$x := n$

$$e^n > \frac{n^n}{n!}$$

✓

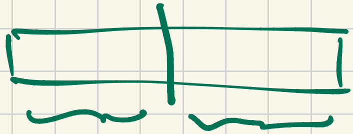
7.25 MERGE(A[1...m], B[1...n])

$\underbrace{A[1\ldots m]}_{\text{sorted}}$, $\underbrace{B[1\ldots n]}_{\text{sorted}}$

```
i, j, k = 0
while i < m ∧ j < n
    if A[i] < B[j] then
        C[k] := A[i]
        i++
    else C[k] := B[j]
        j++
    k++
while i < m
    C[k] := A[i]
    i++      k++
while j < n
    C[k] := B[j]
    j++      k++
```

#2

7.37a MERGE-SORT



$R(n)$: # comparisons

$$R(n) \le R\left(\left\lceil \frac{n}{2} \right\rceil\right) + R\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + n-1 \quad , \quad R(1)=0$$

analyse for $n=2^k$

use method of reverse inequalities:

find $g(n) \ge g\left(\left\lceil \frac{n}{2} \right\rceil\right) + g\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + n-1$, $g(1) \ge 0$

$n=2^k$ if — " — — " —

then $(\forall n=2^k)(g(n) \ge R(n))$

Guess $g(n) := n \log_2 n$ $\qquad g(1)=0$

NTS $\quad n \log_2 n \overset{?}{\ge} 2 \cdot \frac{n}{2} \cdot \log_2 \frac{n}{2} + n-1$

$\quad n \log_2 n \overset{?}{\ge} n(\log_2 n - 1) + (n-1) = n \log_2 n - 1$

DO

12 coins incl 1 fake

genuine coins: equal weight

fake: heavier or lighter

Decide in 3 measurements on a scale:

which coin is fake; is it H or L

=   >   <

4   4 $\boxed{4}$

↑ fake left out

if =

if >

either in left tray, heavy

or in right tray, light

if < (same)

good - 2 $\boxed{2}$ - unknown

if = X

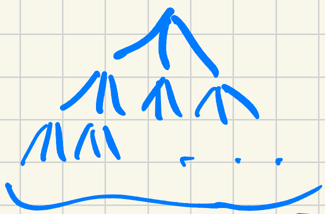
3   $\boxed{3}$

>

can't do in 3 measurements with n coins

(a) $n = 14$: 28 possible outcomes

but



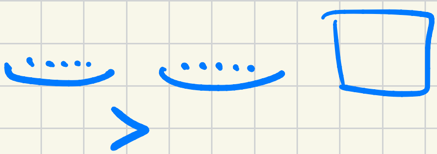
27 leaves

information theory lower bound

(b) $n = 13$:

first measurement: k coins in each tray

if $k \geq 5$



≥ 10 cases left, but only 2 meas. left

≤ 9 cases

if $k \leq 4$

≥ 5 left out

still ≥ 10 cases

Strassen $\quad n = 2^k$

(a1) $M(n) \le 7 \cdot M(\frac{n}{2}) \qquad M(1) = 1$

Claim : $M(n) \le 7^k$

obvious by induction on $\underline{k}$

$$7^k = (2^{\log_2 7})^k = (2^k)^{\log_2 7} = n^{\overbrace{\log_2 7}^{\beta}}$$

$M(n) \le n^\beta$

(a2) $(\forall n = 2^k)\left((a1) \text{ conditions } \not\Rightarrow \underline{M(n) < n^\beta}\right)$

NOT ABOUT ALGORITHM

Proof: we need to give a counterexample

$M(n) := n^\beta = 7^k$

for this $M(n) = 7 \cdot M(\frac{n}{2}),\ M(1) = 1$

LOGIC

(b1) total # op's incl. addition - subtraction + bookeeping

$T(n) \le 7 \cdot T(\frac{n}{2}) + O(n^2)$ $\qquad O(n^{2.3})$

(b2) $T(n) \le 7 \cdot T\left(\frac{n}{2}\right) + O(n^2)$

C

$\Downarrow$

$T(n) = O(n^\beta)$

Pf by reverse ineq.

guess $g(n) = An^\beta + Bn^2 \cdot O(n^\beta)$

$n \ge 2$ $g(n) \ge 7 \cdot g\left(\frac{n}{2}\right) + Cn^2$, $g(1) \ge T(1)$

$n = 1$ $\boxed{A + B \ge T(1)}$

$n \ge 2$ $An^\beta + Bn^2 \ge 7 \cdot \left(A\left(\frac{n}{2}\right)^\beta + B\left(\frac{n}{2}\right)^2\right) + Cn^2$

$B \ge \frac{7}{4}B + C$ $2^\beta = 7$

$-\frac{3}{4}B \ge C$

$\boxed{B \le -\frac{4C}{3}}$

$B := -\frac{4C}{3}$

$A := T(1) + \frac{4C}{3}$

$\forall n$

$g(n) \ge T(n)$

$\therefore T(n) = O(n^\beta)$