HONORS ALGORITHMS

2025-02-07

$n \log n$ — "nearly linear"

linear $O(n)$

thm. $\lim_{x\to\infty} \frac{\ln x}{x} = 0$

$\frac{1/x}{1} = \frac{1}{x} \to 0$

Pf. L'Hôpital's rule

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

assuming: $f(x) \to \infty$, $g(x) \to \infty$, $\exists \lim$ RHS

~~$\exists \lim$ LHS~~

consequence not assumption

assumption

P2

** $\lim \frac{\ln x}{x} = 0$

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = 0$$

$\forall \varepsilon > 0$

$$\lim_{x\to\infty} \frac{\ln x}{x^{\varepsilon}} = 0$$

*

$\lim \frac{\ln x}{x}$ is 0

$\frac{\ln x}{x}$ goes to $\to 0$

~~$\lim \frac{\ln x}{x}$ goes to 0~~

e.g. $\ln x = o\left(x^{\frac{1}{100}}\right)$

Pf 1 L'Hôpital

$$\frac{\frac{1}{x}}{\varepsilon x^{\varepsilon-1}} = \frac{1}{x}\cdot\frac{1}{\varepsilon}\cdot x^{1-\varepsilon} = \frac{1}{\varepsilon}\cdot\frac{1}{x^{\varepsilon}} \to 0$$

$$\therefore \lim \frac{\ln x}{x^{\varepsilon}} = 0$$

Pf 2 substitute new variable $y = x^{\varepsilon} \to \infty$

BOOTSTRAPPING

$\ln y = \varepsilon \ln x$

$$\frac{\ln x}{x^{\varepsilon}} = \frac{\frac{1}{\varepsilon}\ln y}{y} = \underbrace{\frac{1}{\varepsilon}}_{\text{const}} \cdot \underbrace{\frac{\ln y}{y}}_{\to 0} \to 0$$

BOOTSTRAPPING

use initial result $\frac{x}{\ln x} \to 0$

to prove STRONGER result:

$$\frac{(\ln x)^{100}}{x} \to 0$$

Pf $\left(\frac{\ln x}{x^{1/100}} \to 0 \right)^{100\text{th power}}$

DO $\forall N, \varepsilon > 0$ $\quad \frac{(\ln x)^N}{x^\varepsilon} \to 0$

DO same with $\log_2 x = \Theta(\ln x)$

NOTATION

$$e^{x} = \exp(x)$$

$$e^{\sqrt{x^3+x}} = \exp\left(\sqrt{x^3+x}\right)$$

LaTeX

\log

\ln

\exp

\cos

\sin

DEF

$f(n)$ is polynomially bounded

if $(\exists c)\left(f(n) = O(n^c)\right)$

$\in \mathbb{R}, c>0$

$c = 2$ quadratic

$c = 3$ cubic

$n^{3/2}$

Q is $n \log n$ polynomially bounded?

no best c

$$n \log n = O(n^c) \iff c > 1$$

$\boxed{n = \text{bit length of input}}$

polynomial-time algorithm

runs in time $\leq \text{poly}(n) = O(n^c)$

EX multiplication of n-digit numbers

schoolbook alg. is quadratic

Does Dijkstra run in poly. time?

$\boxed{\text{NO}}$ real numbers have no bitlength

Assume all weights rational

$$\frac{a}{b} < \frac{x}{y} \quad \text{if} \quad ay < bx$$

$$\frac{a}{b} + \frac{x}{y} = \frac{ay + bx}{by}$$

$\Downarrow$

Dijkstra poly time

$b(N)$: bitlength of pos integer N

$$= \lceil \log_2 (N+1) \rceil$$

$$b(N_1 \ldots N_k) \simeq \sum b(N_i)$$

if we have input $N_1, \ldots, N_k$

$$b(N_1 \ldots N_k) := \sum b(N_i)$$

$$b(2^N) = \lceil \log_2 (2^N + 1) \rceil = \underline{N+1}$$

Input N length $n = \lceil \log (N+1) \rceil$

Output 2^N $\sim \log N$

$$2^{n-1} \leq N < 2^n$$

Can we compute this in poly time?

NO

length of output

$$N+1 > 2^{n-1}$$

output exponentially long compared length of input