HONORS ALGORITHMS

2025-02-17



$x, \; 2^x, \; 2^{2^x}, \; 2^{2^{2^x}}, \; \ldots$

$a_0 = x \qquad a_{k+1} = 2^{a_k}$

take $x = 2$

$a_k = \left.2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}\right\} k \text{ times}$

$\text{TOWER}_2(k)$

$\text{TOWER}_a(k) = a^{a^{\cdot^{\cdot^{\cdot^{a}}}}}$

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$\log^*(z) = \{\min k \mid \text{TOWER}_2(k) \geq z\}$

$\log^* x \to \infty$

$\text{TOWER}_2(0) = 1$

$\text{TOWER}_2(1) = 2$

(2) 4

(3) 16

(4) $2^{16} = 65{,}536$

(5) $2^{65{,}536} = M$

$\log^* 1 = 0$

$\log^* 2 = 1$

$\log^* 3 = \log^* 4 = 2$

$\log^* 5 = \ldots = \log^* 16 = 3$

$\log^* 17 = \ldots = \log^* 65{,}536 = 4$

$\log^* 65{,}537 = \ldots = \log^* M = 5$

$a + b$ (1)

$a \cdot b = \underbrace{a + \cdots + a}_{b \text{ times}}$ (2)

$a^b = \underbrace{a \cdot \cdots \cdot a}_{b \text{ times}}$ (3)

$\text{TOWER}_a(b) = a^{a^{\cdot^{\cdot^{\cdot^{a}}}}} \Big\} b \text{ times}$ (4)

(5)

$A(k, a, b) : a \; k \; b$

k-th operation

Ackerman's function

| | |
|---|---|
| $2+2=4$ | $2+3=5$ |
| $2 \cdot 2 = 4$ | $2 \cdot 3 = 6$ |
| $2^2 = 4$ | $2^3 = 8$ |
| $\text{TOWER}_2(2) = 4$ | $\text{TOWER}_2(3) = 2^{2^2} = 2^4 = 16$ |
| $\vdots$ | $2 \; 5 \; 3 = \text{TOWER}_2(16) = 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}} \Big\} 16 \text{ times}$ |
| | far more than #particles in Universe |

$A(k, 2, k)$

$\alpha(x) \to \infty$

$\alpha(x) = \min\{k \mid A(k, 2, k) \geq x\}$

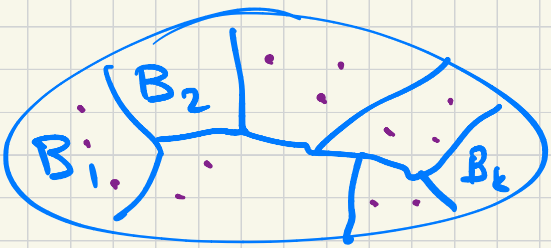
# UNION-FIND DATA STRUCTURE

stores a partition of a given set $\underbrace{\phantom{given\ set}}_{\Omega}$

$$\Omega = B_1 \sqcup \dots \sqcup B_k$$

$B_i \neq \emptyset$

$(\forall i \neq j)\,(B_i \cap B_j = \emptyset)$




QUERIES SERVED:

UNION$(i,j)$ replaces $B_i, B_j$ by $B_i \cup B_j$

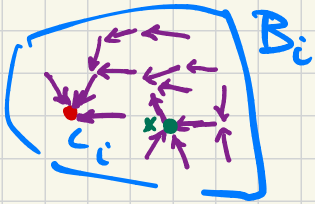
FIND$(x) = i$ if $x \in B_i$
$x \in \Omega$

# hierarchical implementation of UNION-FIND

∀ country has a capital $c_i \in B_i$

$c_i$ is the root of a rooted tree in $B_i$

parent links




FIND(x) : trace parent links from $x \in \Omega$ to capital $c_i$

task : keep these trees shallow

UNION(i,j) : add new link between $c_i$ and $c_j$

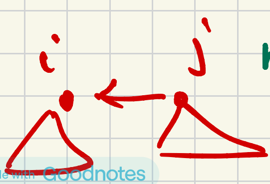
UNION requires deciding which of the two capitals will dominate (win)

decision: $c_i \leftarrow c_j$

OR $c_i \rightarrow c_j$

TWO RULES :

1. biggest wins $O(1)$
2. deepest wins $O(1)$




newdepth $= \max(\text{depth}_i, 1 + \text{depth}_j)$

$i \leftarrow j$

EX each rule guarantees

$\text{depth}_i \le \log_2 n \qquad n = |B_i|$

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sequence of queries

$Q_1 \; Q_2 \; \ldots \; Q_{m+n-1}$

$Q_r$ : FIND($x_r$) or UNION($i_r$, $j_r$)

FIND($y_r$) FIND($z_r$)

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$n-1$ UNIONs

$m$ FINDs

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our current guarantee: $O(n + m \log n)$

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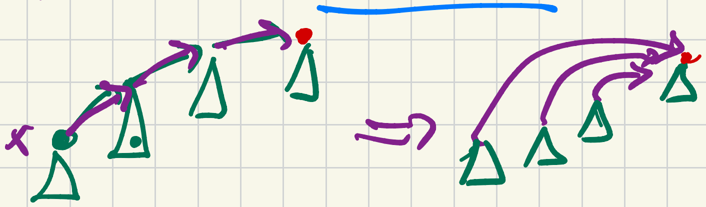
FIND(x) COLLAPSE



# complexity of hierarchical UNION-FIND with collapse

$O(n + m \log\log n)$ ... $O(n + m \cdot \log^* n)$

WAS OPEN: $O(n+m)$ ?

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$$\Theta\left(n + m \cdot \alpha(n)\right)$$

meaning $O(n + m \cdot \alpha(n))$ for all query sequences

and $\forall n$ $\exists$ query sequences with $m = \Theta(n)$ s.t. $\Omega(n + m \cdot \alpha(n))$ are required

more detailed result uses $\alpha(n, m)$ inverse of 2-parameter Ackerman fctn

INPUT: $(V, E, w)$

KRUSKAL

$(V, E)$ graph connected $\quad w : E \to \mathbb{R}$

produces min-weight sp. trees

$\boxed{p\,7}$

we proved this assuming ~~non-negative weights~~

not required

all weights distinct

$\Downarrow$

min weight sp tree is unique

pf of optimality for arbitrary weights

method: PERTURBATON

$$w(e_1) \le w(e_2) \le \cdots \le w(e_m)$$

$$w'(e_j) := w(e_j) + \varepsilon_j$$

$$0 < \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_m$$

$$w'(e_1) < w'(e_2) < \cdots$$

Let $\varepsilon_m := \dfrac{\delta}{3n}$

$$\delta := \min\left( |w(F) - w(F')| : \begin{array}{l} F, F' \text{ sp. trees} \\ \text{s.t. } w(F) \ne w(F') \end{array} \right)$$

$\delta > 0$ by def