HONORS ALGORITHMS

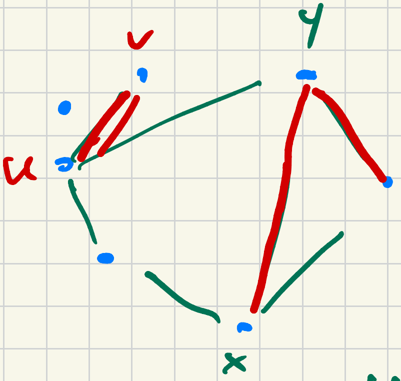
2025-02-19

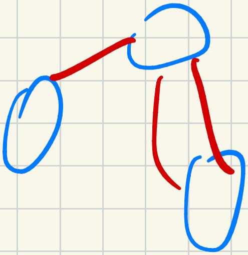
p1

OTAKAR BORŮVKA	1926
VOJTĚCH JARNIK	1930
JOSEPH KRUSKAL	1956
ROBERT PRIM	1957
EDSGER DIJKSTRA	1959
GEORGE SOLLIN	1965

minimum-weight spanning tree algorithms

BORŮVKA



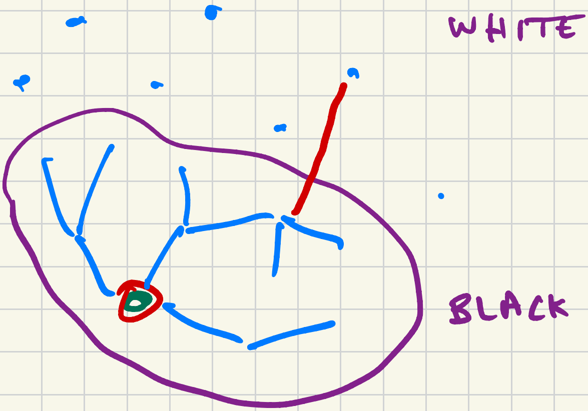


for $v \in V$ select min-weight edge from v

LEMMA
no cycle

JARNIK - PRIM

grow tree from one vertex



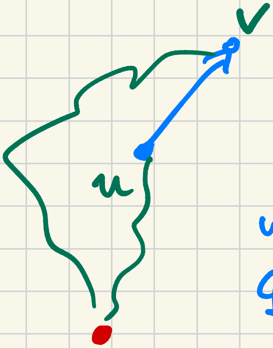
implementation

run Dijkstra
(select a root)

modify RELAX

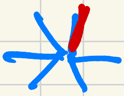
HW

	min sp tree	Dijkstra
INPUT	graph	digraph
	any weights	≥0 weig
	no root	root



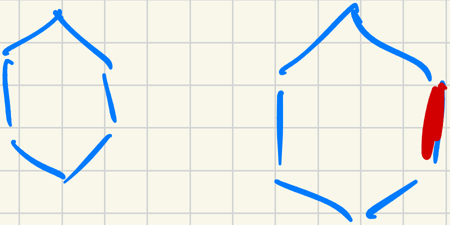
compare
$g(v)$
with
$g(u)+w(u,v)$

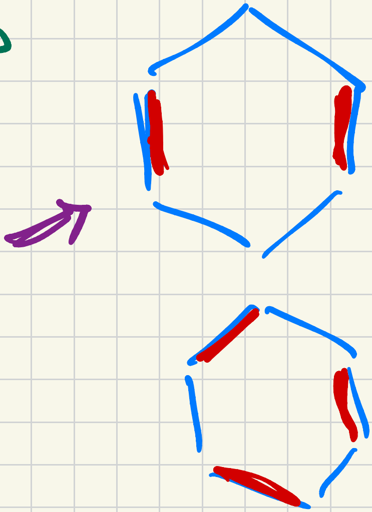
matching in a graph:



$\nu = 1$

a set of disjoint edges





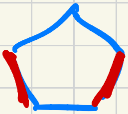
matching number

$\nu(G)$ = max size of a matching

↑

\nu

$$\nu(C_n) = \lfloor \frac{n}{2} \rfloor$$



$$\nu(P_n) = \lfloor \frac{n}{2} \rfloor$$

$P_n \subseteq C_n$

$\nu(K_n) = \lfloor \frac{n}{2} \rfloor$

$\therefore \nu(P_n) \leq \nu(C_n)$

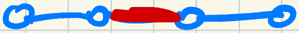
$(\forall G)(\nu(G) \leq \lfloor \frac{n}{2} \rfloor)$

maximal M : if $M \subseteq \overset{\overset{\text{matching}}{\downarrow}}{R}$ then $M = R$

maximum matching $M \subseteq E$ $|M| = \nu(G)$

if M maxim$\underline{\text{um}}$ $\Longrightarrow$ M is maxim$\underline{\text{al}}$

maximal but not maximum matching



HW If M is maximal then $|M| \geq \frac{\nu(G)}{2}$

Greedy matching find <u>maximal</u>

$\downarrow$

$\frac{1}{2}$ - OPTIMAL

DÉNES KÖNIG ~1930

maximum matching in bipartite graphs

poly. time

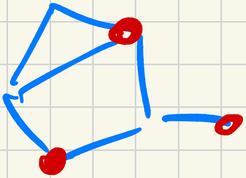
JACK EDMONDS 1965 all graphs

Lp 6

G graph $=(V,E)$

INDEPENDENT SET $W \subseteq V$:

no edges among W



INDEPENDENCE NUMBER

$$\alpha(G) = \max\{|W| : W \text{ indep}\}$$

`\alpha`

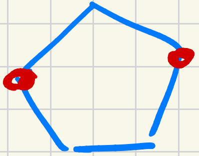
CLIQUE: complete subgraph

W is a clique in $G \iff W$ is indep. in $\overline{G}$

CLIQUE NUMBER

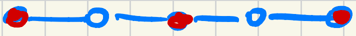
$$\omega(G) = \max\{|W| : W \text{ clique}\}$$

$$\alpha(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$$



$$\alpha(P_n) = \left\lceil \frac{n}{2} \right\rceil$$

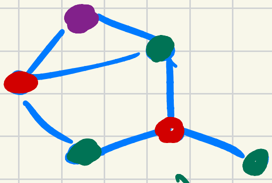
$$\alpha(K_n) = 1$$



coloring (of vertices) : $f: V \longrightarrow \mathbb{N}$

legal coloring :

$$x \sim y \Longrightarrow f(x) \neq f(y)$$

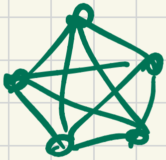


chromatic number:

$$\chi(G) = \min\{k \mid k \text{ colors suffice}\}$$

\chi

$$\chi(K_n) = n$$



$\forall G$

$$1 \leq \chi(G) \leq n$$

$$\chi(G) = 1 \iff E = \emptyset : \overline{K_n}$$

$$\chi(G) \leq 2 \iff \text{bipartite}$$

DO | solve this in lin time

find 2-col OR
find odd cycle

HW $\chi(G) \cdot \alpha(G) \geq n$

Sys. linear equations

m eqs in n unknowns $Ax = b$

INPUT $m \times (n+1)$ matrix $A' = [A \mid b]$

Solve by Gaussian elimination:

$O(mn^2)$ arithmetics operations on reals

Assume all entries are INTEGERS.

Is Gaussian elimination poly time?

CONCERN: blow-up of the numbers involved.

When we add two fractions,

denominators MULTIPLY: $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$

n multiplications can result in numbers with exponentially many digits

```
$x = 2$
repeat n times
    $x := x^2$
return x
```

$x = 2^{2^n}$

#bits: 2^n

JACK EDMONDS 1964

introduced "polynomial time"

Consider sys of lin eq's with integer coeff's

THM (Edmonds) Gaussian elimination (using row operations only) with integer input coeff's runs in polynomial time

CH.

Lemma $A \in \mathbb{Z}^{n \times n}$ $n \times n$ integer matrix

$$b\big(|\det(A)|\big) \le b(A)$$

bit length

COR. let $Ax = b$ be a system of n linear equations in n unknowns. Assume $\det A \neq 0$. Then $\exists!$ solution $z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$

Then $\forall i$

$$b(z_i) \le 2 \cdot b(A') \qquad A' = [A \mid b]$$

System of linear equations

$Ax = b$

$A : m \times n$

$x : n \times 1$

$b : m \times 1$

] integer matrices

$A' = [A|b]$

$m \times (n+1)$

Let us perform GAUSSIAN ELIMINATION (row operations only): $B_0 = A'$, $B_1, \ldots, B_k$ sequence of matrices produced.

LEMMA (Edmonds) Let $\frac{r}{s}$ be an entry of one o the B_i, reduced to smallest terms, i.e., $\gcd(r,s) = 1$. Then $\exists$ square submatrices C, C' of A' such that

$$r \mid \det(C) \neq 0 \quad \text{and} \quad s \mid \det(C') \neq 0$$

COROLLARY (Edmonds)

GAUSSIAN ELIMINATION polynomial time on integer matrices