HONORS ALGORITHMS

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Σ_1, Σ, finite alphabets

P - program   Python $P \in \Sigma_1^*$

X - input $X \in \Sigma^*$

$$P[X] = \begin{cases} \infty & \text{if it does not halt} \\ \text{output} & \text{o/w} \end{cases}$$

<u>P is a halting program</u> if $(\forall X)(P(X) \neq \infty)$

language $L \subseteq \Sigma^*$

L is <u>computable</u> if $\exists P$ s.t.

$$(\forall X \in \Sigma^*)\left(P(X) = \begin{cases} 1 & \text{if } X \in L \\ 0 & \text{o/w} \end{cases} \right)$$

halting program

\# finite strings, such as programs is countably infinite

\# languages is uncountable

$\therefore$ most lang. are not computable

computable language

recursive language ← GÖDEL

set of computable languages $\mathcal{R}$

$L \subseteq \Sigma^*$ is computably enumerable
recursively — " — $\mathcal{RE}$
if $\exists$ program P s.t.

$$(\forall x \in \Sigma^*)\left(\begin{array}{ll} P(x) \neq \infty & \text{if } x \in L \\ P(x) = \infty & \text{if } x \notin L \end{array} \right)$$

$$\mathcal{R} \subseteq \mathcal{RE}$$

Pf $L \in \mathcal{R}$ $P(x) = \begin{cases} 1 & \text{if } x \in L \\ 0 & \text{o/w} \end{cases}$

program Q:
if $P(x) = 1$ halt output 1
else never halt

$x := 0$
$x := x+1$

EX $L \in RE \iff$

either $L = \emptyset$

or $\exists \underline{P}$ always halts

$L = \{ \underline{P}[X] \mid X \in \Sigma^* \}$

$\text{HALTING} = \{ P \mid$ on $\emptyset$ input P halts $\}$

$\text{HALTING} \in RE$

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for all programs $\underline{P}$

run $\underline{P}$

THM $\text{HALTING} \notin R$

"Halting is undecidable"

" is verifiable

non-halting is not "

<u>ALAN TURING</u>

<u>THM</u> HALTING $\notin \mathcal{R}$

Pf by contradiction

Suppose <u>HALTING $\in \mathcal{R}$</u>

Let Q be a corresp. program

Q is a halting program

$\forall$ program P

$$Q(P) = \begin{cases} 1 & \text{if } P \text{ halts on } \emptyset \text{ input} \\ 0 & \text{o/w} \end{cases}$$

Def program B

$$B(X) = \begin{cases} 1 & \text{if } Q(X[X]) = 0 \\ \infty & \text{if } Q(X[X]) = 1 \end{cases}$$

[i.e. $(\forall X)\big(B(X)$ halts $\Longleftrightarrow X[X]$ does not halt$\big)$

let $X = B$

$B(B)$ halts $\Longleftrightarrow B(B)$ does not halt

<u>diagonal method</u>

$\underline{P[X]}$

mimic GEORG CANTOR's proof: $\nexists$ bijection $\mathbb{Z} \leftrightarrow \mathbb{R}$

Epimenides paradox

Greek island CRETE

"ALL CRETANS LIE"

-says Epimenides, a Cretan

If HALTING $\propto$ L

<u>then</u> L is also undecidable

$\propto$ many-one reduction

$$L_i \subseteq \Sigma_i^*$$

$$L_1 \propto L_2$$

$$f : \Sigma_1^* \to \Sigma_2^*$$

s.t. f is computable (by a halting machine)

$f \notin R$ because it is not a predicate

and $(\forall x \in \Sigma_1^*)(x \in L_1 \Leftrightarrow f(x) \in L_2)$

SOLVABILITY of DIOPHANTINE EQUATIONS

example "$x^2+y^2=z^3+5$"

looking for integer solutions

HILBERT in 1900 stated 23 (?) problem

$$coRE = \{\Sigma^* \setminus L \mid L \in RE\}$$

$$coR = R$$

$$R \subsetneq RE$$

$$R = coR \subsetneq coRE$$

$$R \subseteq RE \cap coRE$$

EX $R = RE \cap coRE$

P : set of poly-time computable languages

NP : poly-time verifiable ...

membership has a poly-length "proof"

"witness"

example graph 3-colorability 3COL

NP examples :

3-COL witness of YES answer : 3-coloring

COMPOSITENESS witness: proper divisor

$n \geq 2$, n not prime

CLIQUE:
INPUT: G graph, $k \in \mathbb{N}$
QUESTION: does G contain K_k ?

$$P \subseteq NP \cap coNP$$

CONJ. $\neq$

candidate:

FACT

given x, y does x have a divisor $\leq y$
≥ 2

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in NP triv

in coNP nontriv

in P ✗ ← belief