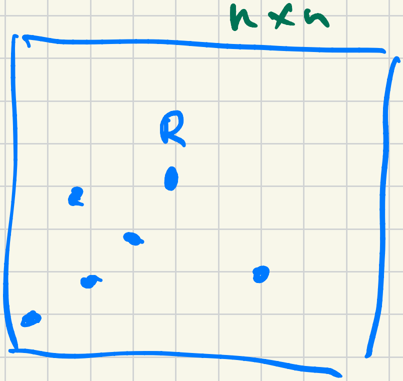
ALGORITHMS PROBLEM SESSION

2025-02-28

15.57 (d) Car race $|R| < n$



phase space

$\Phi = R \times \{velocities\}$

(x,y) (u,v)

$|\Phi| = O(|R| \cdot n)$

$(x_1,y_1) \to (x_2, y_2)$

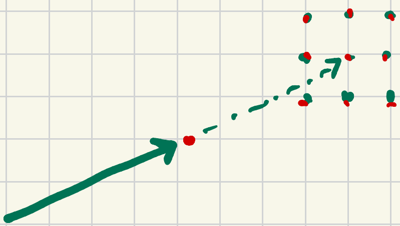
$\underline{x}_1$ $\underline{x}_2$

velocity $\underline{x}_2 - \underline{x}_1 = \underline{v}_2$

$\| v_1 - v_2 \|_{max} \leq 1$

difference in coordinates ≤ 1

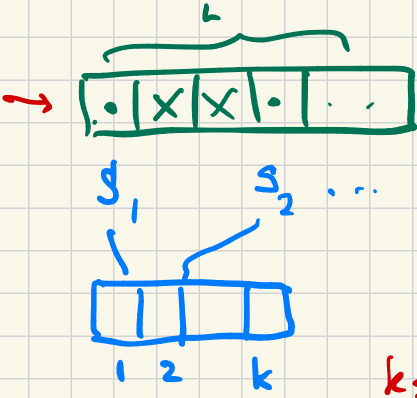
$\| v_i \|_{max} = O(\sqrt{n})$



$$\underbrace{\to \longrightarrow \xrightarrow{k}}_{\frac{k(k+1)}{2} \leq n}$$

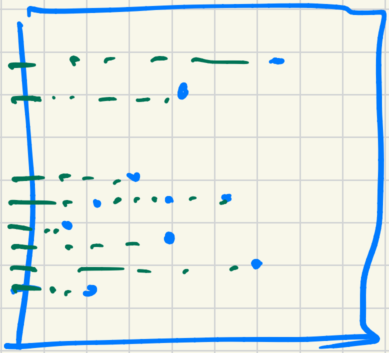
$k < \sqrt{2n}$

occupied rows



$k \le |R|$

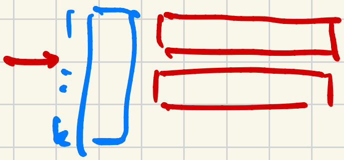
(p2



$k \times n$

R : list $x_1 \dots x_r$

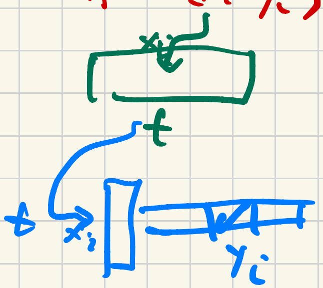
$O(|R| + n)$



$k \times n$ array

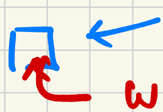
single pass through R

if $(x_i\ y_i)$



O(1) lookup
of location

BFS



$(x_i\ y_i) \in R$

WGR

p3

integer KNAPSACK

bit length of input

DP not poly.time

$$N = \sum_{i=1}^{n} \left(b(w_i) + b(v_i)\right) + b(W)$$

WLOG $w_i \le W$

identify hard cases for DP $\Theta(n \cdot W)$ arithm. op's

Case: W is large

n-bit $\Omega(n \cdot 2^n)$ "

lower bd on complexity

upper bd on input length

$v_i := 1$

$w_i \le W$

$N \le 2n \cdot n = 2n^2 = O(n^2)$

weights
values: $+n$
weight lim

$n = \Omega(\sqrt{N})$

cost $> 2^n > 2^{\sqrt{N}}$

growing faster than any polynomial

19.41 (a) greedy matching produces <u>maximal</u> matching

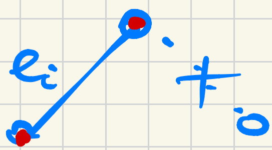
$e_1 \quad e_2 \quad \overset{\checkmark}{e_j} \ \ldots \ e_m$

$M = \emptyset$ —— initialize matching

for $i = 1$ **to** m

  if e_i $\underbrace{\text{can be added}}_{\text{vertices of } e_i \text{ have not been visited}}$ then add e_i to M

  mark the vertices of e_i "visited"

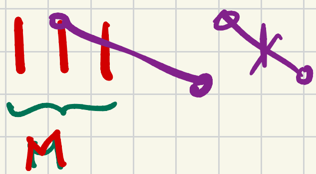


Pf Suppose M misses an edge e_j

but then e_j should have been added in j^{th} round

(b) Claim: if M is a maximal matching then $\underbrace{\nu(G)}_{\text{size of maximum matching}} \le 2|M|$

Pf



covers $2|M|$ vertices

maximum matching R

$\forall x \in V(M)$ at most one edge in R hits x

$\therefore \quad \nu(G) = |R| \le |V(M)| = 2 \cdot |M|$

19.68

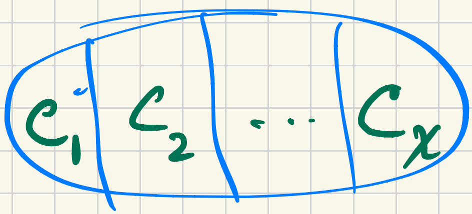
$$\alpha(G)\cdot\chi(G)\geq n$$

independence number: max size of an independent set

↑ no edges among these vertices

chromatic number: min # colors for a legal coloring

fix an optimal coloring



C_i = set of vertices of color i

C_i is indep. ← observation

$\therefore\ |C_i|\leq\alpha$

$\therefore\ n=\sum|C_i|\leq\chi\cdot\alpha$ ✓

$\omega \le \chi$

$\boxed{\text{case } \omega < \chi}$

19.74 smallest G s.t.

clique number $\omega(G) = 3$ so $K_4 \not\subseteq G$

and $\chi(G) = 4$

what matters: ≥ 4

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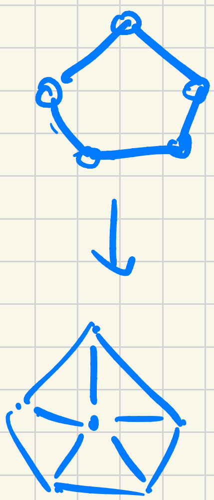
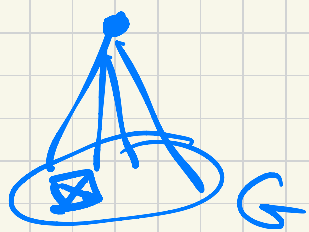
thinking: $\omega \le 2$ no triangles

$\chi \ge 3$ not bipartite





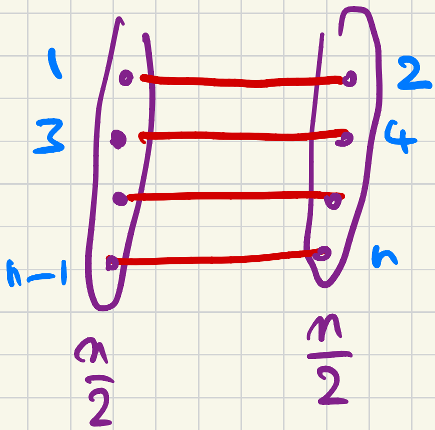
ω grows by 1

χ " by 1

19.91 find bipartite graph s.t. greedy coloring uses $\frac{n}{2}$ colors

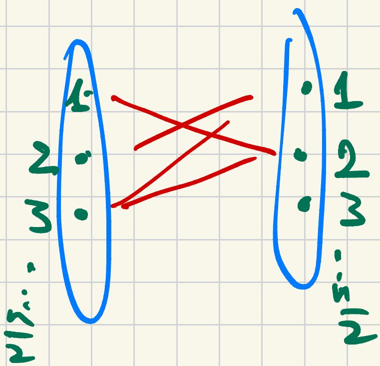
n even with ordered set of vertices

$\boxed{p8}$



$deg \geq \frac{n}{2} - 1$

Colors

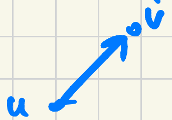


19.20 JARNIK-PRIM: min sp tree

implem via DIJKSTRA ← Turing award

1. declare source
2. modify RELAX

DIJKSTRA



if $g(v) > g(u) + w(u,v)$

then $g(v) := \ldots$

$p(v) := u$

JARNIK

if $g(v) > w(u,v)$

then $g(v) := \ldots$

$p(v) := u$