HONORS ALGORITHMS | 2025-03-05

P1

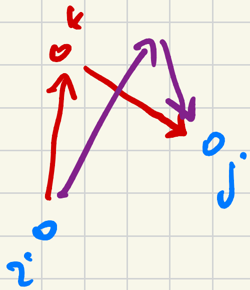
Adjacency matrix of a digraph

$V = [n] \qquad G = (V, E)$

$A_G = (a_{ij}) \qquad a_{ij} = \begin{cases} 1 & \text{if } ij \in E \\ 0 & \text{o/w} \end{cases}$

$n \times n$

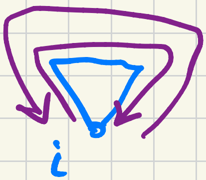
$A_G^2 = (b_{ij}) \qquad b_{ij} = \sum_{k=1}^{n} \underbrace{a_{ik} a_{kj}}_{} =$



2-step walks from i to j

EX $A_G^t = (a_{ij}^{(t)})$ $\quad a_{ij}^{(t)}$ counts t-step walks from i to j

G graph



$(A_G^3)_{ii} = 2$ #triangles $\underline{\text{at}}$ vx i

times

$6 \cdot$ #triangles in G $= \mathrm{tr}(A_G^3)$

trace $= \sum$ diagonal entries

We can count triangles in $O(n^{\log_2 7})$ using Strassen's matrix multipl.

fixed matrix

$A \cdot \underline{x}_1$ $\quad\quad \Theta(n^2)$

$n \times n$

$A \cdot \underline{x}_2$ $\quad\quad O(n^2)$

$\vdots$

$A \cdot \underline{x}_n$ $\quad\quad O(n^2)$

$$\overline{\quad O(n^3) \quad}$$

$X = [\underline{x}_1 \ \ldots \ \underline{x}_n]$

$n \times n$

$A \cdot X = [A x_1 \ \ldots \ A x_n]$

Strassen

$P = \{$ poly·time languages $\}$

↑ decision problem

$NP = \{$ poly·time time verifiable lang. $\}$

$L \subseteq \Sigma^*$

$L \in NP$ if $\exists\, c,\ L_1 \in P$ s.t.

↑ judge

$$(\forall x \in \Sigma^*)(x \in L \iff (\exists w \in \Sigma_1^*)(|w| \le |x|^c \wedge (x,w) \in L_1)\}$$

witness ("proof")

NP-definition

Karp. reduction reducing L_1 to L_2

$$f: \Sigma_1^* \longrightarrow \Sigma_2^* \qquad L_i \subseteq \Sigma_i^*$$

(1) f poly-time computable

(2) $\underbrace{(\forall x \in \Sigma_1^*)}_{\uparrow}\left(x \in L_1 \longleftrightarrow f(x) \in L_2\right)$

input to membership problem in L_1

$L_1 \propto_{\text{Karp}} L_2$ if $\exists f$: Karp red. from L_1 to L_2

L_1 is Karp-reducible to L_2

DEF L is NP-complete if

$(\forall M \in NP)(M \propto_{\text{Karp}} L)$

$L \in NP$

Toronto Moscow

Stephen Leonid

Cook-Levin Thm 1972

SAT is NP-complete

$\uparrow$

Boolean circuit satisfiability

Given a Boolean ckt $\mathcal{B}$

$$(\exists ?\ \underline{x} \in \{0,1\}^n)(\mathcal{B}(\underline{x}) = 1)$$

COROLLARY If $L \in NP$ s.t.

$SAT \propto_{Karp} L$

then L is NP-complete

NPC = {NP-complete languages}

Thm $\underbrace{3SAT}$ $\in$ NPC

Satisfiability of 3-CNF formulas $\bigwedge_{j=1}^{m} C_j$

$C_j = x_i \vee \bar{x}_j \vee \bar{x}_k$

Richard Karp 1973

reduced ↓ from 3-SAT

3-COL

∃ clique

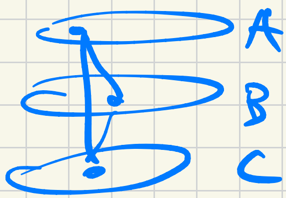
CLIQUE $= \{ (G, k) \mid G \text{ has a clique of size } k \}$

graph, target clique size

HAM

∃H-cycle

→ 3D-MATCHING

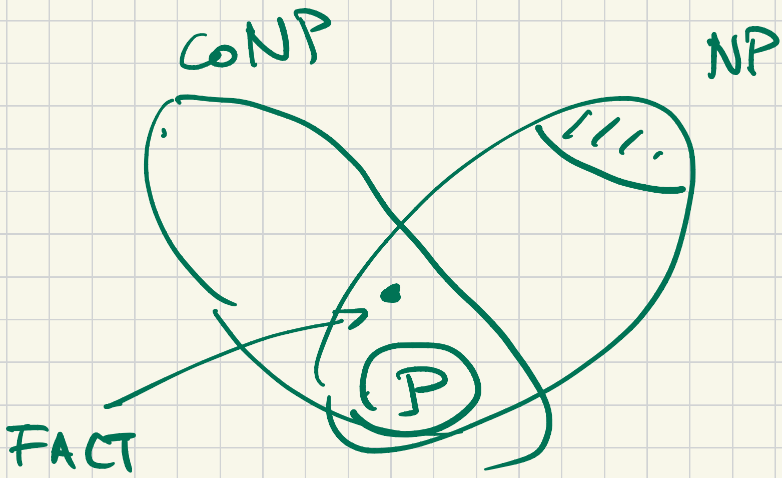


Input: triples $T_i \in A \times B \times C$

question: ∃ set of disj triples that cover $A \cup B \cup C$

If $L \in NPC$ then $FACT \propto_{Karp} L$

$FACT \in NP \cap coNP$



Ex If FACT $\in$ NPC

then $NP = coNP$ ← conjecture $NP \neq coNP$

Ex If $L_1 \propto_{karp} L_2$ and $L_2 \in NP$

then $L_1 \in NP$