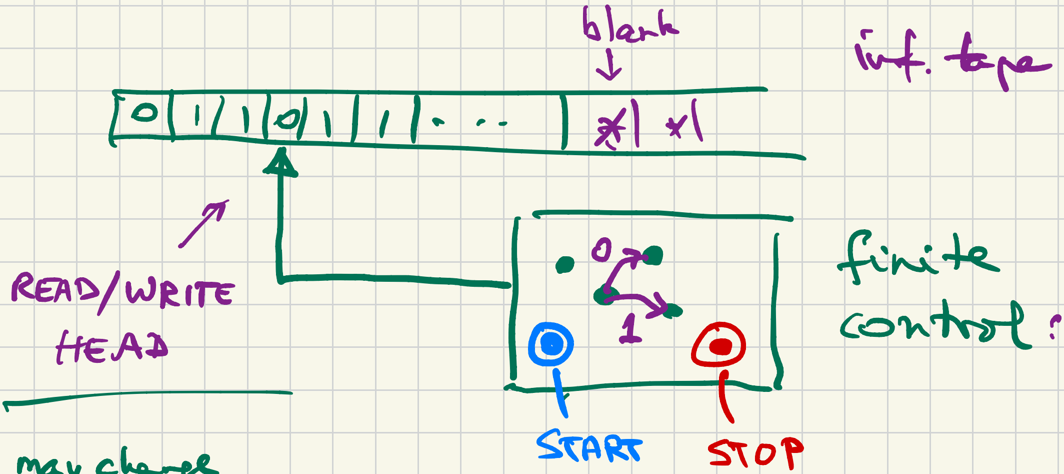
HONORS ALGORITHMS

2025-03-07

P 1

TURING MACHINE



may change cell content

←	|	→
L	0	R

finite #states

transitions

Cook-Levin Theorem

SAT $\in$ NPC = {NP-complete languages}

$\Sigma = \{0,1\}$

$\Sigma^* = \bigcup_{k=0}^{\infty} \Sigma^k$

↑ strings of length k

$L \subseteq \Sigma^*$

$L_n = \Sigma^n \cap L \subseteq \{0,1\}^n$

$\longleftrightarrow$ Boolean fcn on n variables

repr. by ckt C_n

Thm $L \in P \iff$

$\exists$ poly-size circuits C_n size $(C_n) = O(n^c)$

and given n C_n is computable in time $O(n^c)$

Pf $\Leftarrow$ immediate

$\Rightarrow$ use TM's for easy encoding of computation by ckts

$L \in NP \iff \exists\, c,\ M \in P$ ← judge's language

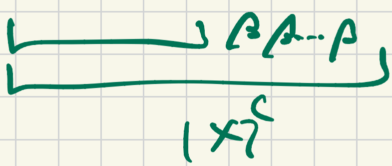
$x \in L \leftrightarrow (\exists w)\big((\underline{|w| \le |x|^c}) \wedge (x, w) \in M\big)$

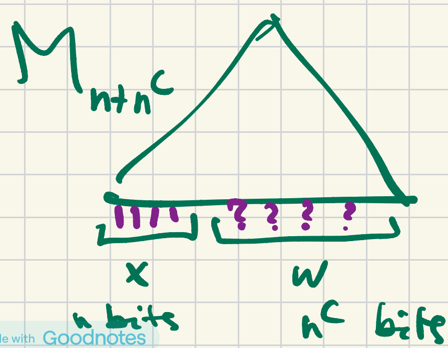
input

M computed by a $\underline{\text{uniform}}$ family of poly-size Boolean circuits

$\underline{\text{circuits}}$ themselves efficiently computable

WLOG $|w| = |x|^c$ add dummy symbols





$\underline{\exists w\ ?}$

SAT

NEXT: 3SAT : 3CNF satisfiability

$\bigwedge C_j$ ← 3-clause

disjunctive

$(x_7 \vee \bar{x}_9 \vee x_{22})$

Claim $\text{SAT} \propto_{\text{Karp}} \text{3SAT}$

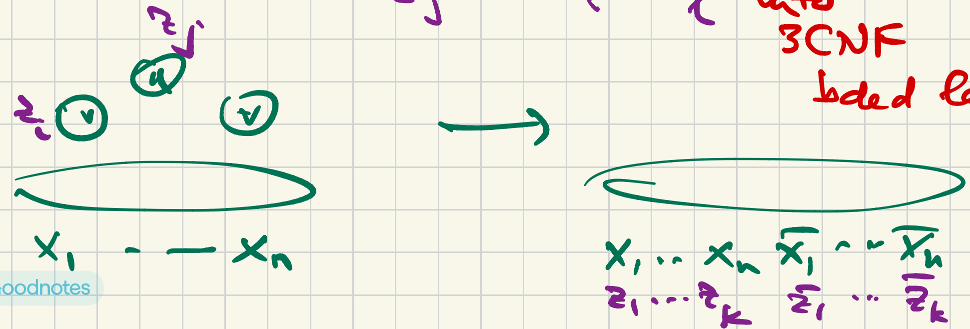
$f : \{\text{Boolean ckts}\} \to \{\text{3CNF formulas}\}$

1. f computable in poly time
2. $(\forall B)(B \in \text{SAT} \iff f(B) \in \text{3SAT})$

B: Boolean ckt input to SAT

WLOG fan-in = 2

$z_j \leftrightarrow z_i \wedge \bar{z}_\ell$ turn this into 3CNF bded length



NEXT:

$$\text{CLIQUE} = \{(G, k) \mid G \supseteq K_k\} \in NP$$

graph

witness: clique of size k

Claim $3SAT \propto_{Karp}$ CLIQUE

$\therefore$ CLIQUE $\in$ NPC

$f : \{3CNFs\} \to \{(G, k)\}$

s.t.

① f is efficiently computable

② $(\forall B)(B \text{ satisfiable} \leftrightarrow \underbrace{(G, k)}_{f(B)} \in \text{CLIQUE})$

3CNF

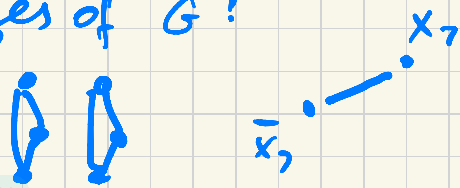
C_1

x_1	$\cancel{x_7}$
$\overline{x_7}$	x_2
x_4	$\overline{x_4}$

m clauses

$\Rightarrow G$ has $3m$ vertices

edges of $\overline{G}$!



If C is a clique in G then $|C| \le m$

Claim: $\exists$ Cliques of size m $\Longleftrightarrow B$ satisfiable

SUBSET-SUM problem $\in NP$ ✓

input: $(a_1, \ldots, a_n, b)$ $\quad a_i, b \in \mathbb{N}$

Question: $\exists ?\ I \subseteq [n]$ s.t.

$$\sum_{i \in I} a_i = b$$

Claim

3-SAT $\propto$ 3DM $\propto$ SUBSET-SUM

3-dim matching

$\therefore$ SUBSET-SUM $\in$ NPC

EX SUBSET-SUM is special case of integer KNAPSACK

$\therefore \in$ NPC

KNAPSACK :

can be approximated within factor of $(1-\varepsilon)$ in time $O\left(\frac{n^3}{\varepsilon}\right)$

PTAS : poly-time approximation scheme

NP. hard: at least as hard as anything in NP

MAX-3SAT

INPUT : 3-clauses $C_1 \cdots C_m$

OPT : max # simult. satisfiable clauses

$\frac{7m}{8}$ always satisfiable $x_7 \vee \bar{x}_9 \vee x_{21}$

$\therefore$ random assignment: $E\left(\#\text{ satisf clauses}\right) = \frac{7m}{8}$

$\therefore$ $\varepsilon = \frac{1}{8}$ approximation

THM. approximation within $\frac{1}{8} - \delta$ is NP-hard $\forall \delta > 0$

MAX 3-SAT APPROXIMATION RESISTANT

proofs of hardness of approximation

based on

theory of interactive proofs /

probabilistically checkable proofs

check a constant number of places
in the proof
accept or find inconsistency

PCP theorem Every proof can be turned
into a PCP in poly. time