The greatest term in Dobiński’s formula

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Date: May 15, 2020

$B_n$, the $n^{\text{th}}$ Bell number, counts the partitions of a set of $n$ elements. Dobiński’s formula (1877) expresses this quantity as a sum of positive terms as follows.

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}.$$

**Question.** Find the greatest term in Dobiński’s formula; let $k_0$ denote the value of $k$ that maximizes $k^n/k!$.

Let $x = \lambda(n)$ be defined by the equation $x \ln x = n$.

**Claim.** $|k_0 - \lambda(n)| < 1$.

In other words, $k_0$ is either $\lfloor \lambda(n) \rfloor$ or $\lceil \lambda(n) \rceil$.

We prepare the proof with an estimate.

**Lemma.**

$$\frac{1}{k} > \ln(k + 1) - \ln(k) > \frac{1}{k+1}.$$

**Proof.** Both inequalities follow from the identity

$$\ln(k + 1) - \ln(k) = \int_k^{k+1} \frac{dt}{t}.$$

**Proof of the Claim.** Let $f(k) = k^n/k!$. Let $\ell = \lceil \lambda(n) \rceil$, so we have $\ell \leq \lambda(n) < \ell + 1$. We shall show that $k_0 \in \{\ell, \ell + 1\}$. This will follow from the following two statements.

(a) If $k \leq \ell - 1$ then $f(k) < f(k + 1)$.

(b) If $k \geq \ell + 1$ then $f(k) > f(k + 1)$.

We need to study the quantity

$$\frac{f(k + 1)}{f(k)} = \frac{(k + 1)^n}{(k + 1)!} \cdot \frac{k!}{k} = \frac{(k + 1)^{n-1}}{k^n}. \quad (1)$$

Let

$$g(k) = \ln \left( \frac{f(k + 1)}{f(k)} \right) = (n-1) \ln(k + 1) - n \ln k. \quad (2)$$

We need to show that

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(a) if \( k \leq \ell - 1 \) then \( g(k) > 0 \), and

(b) if \( k \geq \ell + 1 \) then \( g(k) < 0 \).

**Proof of statement (a).** We restate the inequality \( g(k) > 0 \):

\[
(n - 1) \ln(k + 1) > n \ln k,
\]

or equivalently,

\[
\frac{\ln(k + 1)}{\ln k} > \frac{n}{n - 1}.
\]

Subtracting 1 from each side, this is equivalent to

\[
\frac{\ln(k + 1) - \ln k}{\ln k} > \frac{1}{n - 1}.
\]

By the Lemma, we have \( \ln(k + 1) - \ln k > 1/(k + 1) \), so for Eq. (5) to hold, it suffices if

\[
\frac{1}{(k + 1) \ln k} \geq \frac{1}{n - 1};
\]

in other words,

\[
(k + 1) \ln k \leq n - 1.
\]

So we need to show that if \( k \leq \ell - 1 \) then Eq. (7) holds. But in this case \( (k + 1) \ln k \leq \ell \ln(\ell - 1) = \ell \ln(\ell) - \ell \ln(\ell - 1) < n - \ell (\ln(\ell) - \ln(\ell - 1)) < n - 1 \). In the last step we used that \( \ln(\ell) - \ln(\ell - 1) > 1/\ell \) (see the Lemma). This completes the proof of statement (a).

**Proof of statement (b).** We need to show that if \( k \geq \ell + 1 \) then \( g(k) < 0 \), i.e.,

\[
(n - 1) \ln(k + 1) < n \ln k.
\]

As above (see Eq. (5)), this is equivalent to

\[
\frac{\ln(k + 1) - \ln k}{\ln k} < \frac{1}{n - 1}.
\]

By the Lemma, we have \( \ln(k + 1) - \ln k < 1/k \), so for Eq. (9) to hold, it suffices if

\[
\frac{1}{k \ln k} < \frac{1}{n - 1};
\]

in other words,

\[
k \ln k > n - 1.
\]

In fact, by definition, \( k \ln k \geq (\ell + 1) \ln(\ell + 1) > n \). □