The (0,1)-Integer Linear Programming Problem

Algorithms – CMSC 37000

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A linear inequality in the variables x, y, z is an inequality of the form $3x - 5y + z \leq 6$. (We may replace the coefficients $3, -5, 1, 6$ by arbitrary real numbers, and the number of variables may also be arbitrary. We shall assume, however, that *all coefficients are integers.*) The *linear programming* (LP) problem takes a system of linear inequalities as input (m inequalities in n variables) and asks its *feasibility* (does there exist a solution, satisfying all the given inequalities at the same time?). The *integer linear programming* (ILP) problem asks the existence of a solution in *integers* (each variable must take an integral value). A $(0, 1)$-ILP asks the existence of a solution where each variable takes the value 0 or 1.

LP is one of the most widely applied algorithmic problems.

Fact. LP is solvable in polynomial time (Khachiyan 1979, Karmarkar 1983).

(i) Give an example of an LP (with integer coefficients) which is feasible (has solution(s)) but which is not feasible as an ILP. Use as few variables as possible.

(ii) Same as item (i) but all coefficients in the LP must be 0, 1, or -1, including the numbers on the right hand side.

(iii) Prove that (0,1)-ILP belongs to NP. State the (polynomial-time verifiable) certificate (of the yes-answers). Indicate why it is verifiable in polynomial time (length of input to verification algorithm, estimated running time of verification).

(iv) Is it evident that ILP belongs to NP? Argue your answer. Be as specific about the potentially difficult technical detail as you can.

(v) Assuming 3-COL (3-colorable graphs) is NP-complete, prove that (0,1)-ILP is NP complete, by giving a Karp-reduction from 3-COL to (0,1)-ILP.