Orthogonal polynomials are real-rooted without multiple roots, and they interlace

Abigail Ward's proof, REU 2015
Instructor: László Babai

Document last updated May 7, 2023

Definitions.
Let $w : \mathbb{R} \to \mathbb{R}$ be a real function. We shall call w a **weight function** if the following four conditions hold.

(a) w is Lebesgue measurable

(b) $(\forall x \in \mathbb{R})(w(x) \geq 0)$

(c) $\int_{-\infty}^{\infty} w(x)\,dx > 0$

(d) $(\forall k \geq 0)(\int_{-\infty}^{\infty} x^{2k} w(x)\,dx < \infty)$.

Under these conditions, the formula

$$\langle p, q \rangle = \int_{-\infty}^{\infty} p(x)q(x)w(x)\,dx \tag{1}$$

defines a positive definite inner product on the space $\mathbb{R}[x]$ of real polynomials.

Let $f_0, f_1, \dots$ be a sequence of polynomials such that $\deg(f_n) = n$. Then these polynomials form a basis of $\mathbb{R}[x]$.

We say that the f_n form a sequence of **orthogonal polynomials** with respect to the weight function w if additionally they are pairwise orthogonal with respect to the inner product (1), i. e., for all $i \neq j$, $\langle f_i, f_j \rangle = 0$.

Such a sequence of polynomials can be constructed by applying Gram–Schmidt orthogonalization to the basis $(1, x, x^2, \dots)$ of $\mathbb{R}[x]$.
We now state the result indicated in the title.

Theorem. Let $f_0, f_1, \dots$ be a sequence of orthogonal polynomials with respect to some weight function w. Then, for all $n \geq 0$, the polynomial f_n has n distinct real roots and the roots of f_{n+1} and f_n strictly interlace.

The typical proof of this result proceeds by first proving that every sequence of orthogonal polynomials satisfies a "3-term recurrence" of the form

$$f_n(x) = (\alpha_n x + \beta_n) f_{n-1}(x) - \gamma_n f_{n-2}(x) \tag{2}$$

for suitable real numbers $\alpha_n, \beta_n, \gamma_n$ where $\alpha_n, \gamma_n > 0$, assuming (as we may without loss of generality) that the leading coefficient of each f_n is positive.

History. In a U. Chicago Math REU class in summer 2015 I (LB) assigned the Theorem as a challenge problem. Abigail Ward, then a recent recipient of her Bachelors degree, was a TA in the class. She solved the problem within days. A remarkable aspect of her proof of this classical result is that it does not rely on the 3-term recurrence but goes straight to the proof of the Theorem. This provides the most elegant proof I am aware of of the first statement in the Theorem (that orthogonal polynomials have n distinct real roots), in just 12 lines (below).

The proof presented below follows the outline Ward gave me in a letter on July 16, 2016.

Proof adapted from Abigail Ward, UChicago REU, June 2015.

We first note that for all n, the polynomials $f_0, \ldots, f_{n-1}$ span the n-dimensional vector space of all polynomials of degree at most $n-1$. Since each f_n is orthogonal to each f_i for $0 \le i \le n-1$, f_n is orthogonal to all polynomials of degree less than n.

Lemma 1. Let p be a polynomial of degree $n \ge 0$. Assume p is orthogonal to all polynomials of degree $\le n-2$. Then p has n distinct real roots.

Proof. Obvious for $n = 0, 1$. Let now $n \ge 2$. Assume for a contradiction that p does not have n distinct real roots. Assume without loss of generality that the leading coefficient of p is positive. Let $\lambda_1, \ldots, \lambda_k$ denote those distinct roots of p that have odd multiplicity, i. e., those roots at which p changes sign. Observe that $k \le n-2$. Consider the degree-k polynomial $q = (x-\lambda_1)\cdots(x-\lambda_k)$. Note that pq is everywhere non-negative, so

$$\langle p, q\rangle = \int_{-\infty}^{\infty} p(x)q(x)w(x)\,dx > 0\,. \tag{3}$$

Thus p is not orthogonal to q, which is a polynomial of degree at most $n-2$, a contradiction. □

Corollary. For $n \geq 1$ and any scalar $\sigma \in \mathbb{R}$, the polynomial $f_n - \sigma \cdot f_{n-1}$ has n distinct real roots. In particular, f_n has n distinct real roots.

Proof. Indeed, $f_n - \sigma \cdot f_{n-1}$ is orthogonal to all polynomials of degree $\leq n - 2$. □

Lemma 2. For $n \geq 0$, f_{n+1} and f_n do not share any roots.

Proof. Assume for a contradiction that $f_{n+1}(\zeta) = f_n(\zeta) = 0$ for some $\zeta \in \mathbb{R}$. We know that $f_n'(\zeta) \neq 0$ because f_n has no multiple roots. Let $\sigma = \frac{f_{n+1}'(\zeta)}{f_n'(\zeta)}$ and $h = f_{n+1} - \sigma \cdot f_n$. Then $h(\zeta) = h'(\zeta) = 0$, so ζ is a multiple root of h, contradicting the Corollary. □

We now show that for $n \geq 0$, the roots of f_{n+1} and f_n interlace. This is vacuously true for $n = 0$. Assume now $n \geq 1$. Let $\lambda_0 < \lambda_1 < \cdots < \lambda_n$ be the roots of f_{n+1}.

Assume for a contradiction that the roots of f_{n+1} and f_n do not interlace. This means that f_{n+1} has two consecutive roots, $\lambda_i < \lambda_{i+1}$, with no root of f_n in the closed interval $[\lambda_i, \lambda_{i+1}]$.

Consider the function $g = f_{n+1}/f_n$. This is a differentiable function on the closed interval $[\lambda_i, \lambda_{i+1}]$, and $g(\lambda_i) = g(\lambda_{i+1}) = 0$. By Rolle's Theorem, there exists a point $\zeta \in (\lambda_i, \lambda_{i+1})$ for which $g'(\zeta) = 0$; we then have that $f_{n+1}(\zeta)f_n'(\zeta) = f_{n+1}'(\zeta)f_n(\zeta)$, which implies that

$$g(\zeta) = \frac{f_{n+1}(\zeta)}{f_n(\zeta)} = \frac{f_{n+1}'(\zeta)}{f_n'(\zeta)}. \tag{4}$$

Let $\sigma = g(\zeta)$. Consider now the function $f_{n+1} - \sigma \cdot f_n$. We know by the above that this function vanishes along with its derivative at ζ, thus ζ is a multiple root, contradicting the Corollary. □