Asymptotic Equality and Inequality

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1 Sequences

Notation 1.1. $\mathbb{N} = \{1, 2, 3, \dots\}$ denotes the set of natural numbers (positive integers), and we write $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ for the set of non-negative integers.

Definition 1.2. A **sequence** is a function whose domain is a subset of $\mathbb{N}_0$.

If the domain of our sequence a is the set $I \subseteq \mathbb{N}_0$ then for $i \in I$ we typically write a_i instead of $a(i)$. For the sequence $(a_0, a_1, a_2, \dots)$ we write $(a_n \mid n \in \mathbb{N}_0)$ or simply (a_n). The traditional notation for this sequence is $\{a_n\}$, which, however, risks a confusion between the sequence $(a_n \mid n \in I)$ and the set $\{a_n \mid n \in I\}$: in a sequence, the order of items matters and repeated items count as separate entries; in a set, the order does not matter and repeated entries are ignored. For instance, $(4, 2, 2, 3, 3) \neq (3, 4, 2, 3, 2)$, and these are sequences of length 5, but $\{4, 2, 2, 3, 3\} = \{3, 4, 2, 3, 2\} = \{2, 3, 4\}$ is a set of size 3.

Convention 1.3. In these notes, by a "sequence" we shall always mean an **infinite sequence** (the domain is infinite), unless expressly stated otherwise.

Definition 1.4. We say that the sequence $(b_n \mid n \in J)$ is a **subsequence** of the sequence $(a_n \mid n \in I)$ if $J \subseteq I$ and $(\forall j \in J)(a_j = b_j)$.

Definition 1.5. A **predicate** on a set Ω is a function $P : \Omega \to \{0, 1\}$.

The Boolean values 1 and 0 are often interpreted to mean YES/NO or TRUE/FALSE. If for some $a \in \Omega$ we write "$P(a)$," this means $P(a) = 1$, i.e., "$P(a)$ is TRUE." The case $P(a) = 0$ means $P(a)$ is FALSE; this is also expressed as "$\neg P(a)$," i.e., the negation of $P(a)$ is TRUE.

Example 1.6. If $\Omega = \mathbb{N}_0$ and $P(a)$ means "a is a sum of two squares" (i.e., $(\exists u, v \in \mathbb{N}_0)(a = u^2 + v^2)$) then $P(13)$ and $\neg P(15)$.

Definition 1.7. Let $(a_n \mid n \in I)$ be a sequence that takes values in a set Σ (the codomain of a). Let P be a predicate on Σ. We say that "$P(a_n)$ holds for **all sufficiently large** n" if $(\exists n_0)(\forall n \in I)(n \geq n_0 \Rightarrow P(a_n))$. In this case we also say that P **eventually** holds for a_n, or we say that the sequence (a_n) **is eventually** P.

We refer to n_0 as a **threshold** (beyond which P is guaranteed to hold). If n_0 is a valid threshold an $n_1 \geq n_0$ then n_1 is also a valid threshold.

Example 1.8. The sequence (a_n) is **eventually nonzero** if
$(\exists n_0)(\forall n \in I)(n \geq n_0 \Rightarrow a_n \neq 0)$.

Exercise 1.9. (a) Define what it means for a sequence to be eventually zero.

(b) Find a sequence that is neither eventually zero nor eventually nonzero.

Exercise 1.10. Give a simple sentence in plain English expressing the negation of the statement that "the sequence S is eventually nonzero."
Do not use the word "not" and avoid mathematical usage such as "there exist(s)." You may use the word "zero" and variants of the word "infinity."

We shall deal with **predicates on the set of sequences of real numbers**. So in this case, Ω, the domain of the predicate, will not be a subset of $\mathbb{N}_0$ but the set of all sequences.

Examples 1.11.

1. For a sequence s, let $C(s)$ mean "s is convergent." Then C is a predicate on the sequences of real numbers.
2. For a sequence s, let $L(s)$ mean "s is an ℓ^2-sequence," i. e., for $s = (a_n \mid n \in I)$, $L(s)$ means that $\sum_{i \in I} a_i^2 < \infty$. Again, L is a predicate on the sequences of real numbers.

Definition 1.12. A **tail predicate** is a predicate on sequences that does not change its value if we change a finite number of entries of the sequence. The predicate is not affected even if a finite number of terms of the sequence are *undefined* (like $0/0$); we just view "undefined" as another symbol which will not appear beyond the threshold.

Examples 1.13. of tail predicates:

1. "Convergence" (TRUE for convergent sequences)
2. "Eventually constant." This predicate is TRUE for a sequence $(a_n \mid n \in I)$ if $(\exists n_0 \in \mathbb{N}_0$ and $c \in \mathbb{R})(\forall n \in I)(n \geq n_0 \Rightarrow a_n = c)$.
3. "Eventually monotone nondecreasing." This predicate is TRUE for a sequence $(a_n \mid n \in I)$ if $(\exists n_0)(\forall n, m \in I)(m \geq n \geq n_0 \Rightarrow a_n \leq a_m)$.
4. ℓ^2-sequence.

2 Additional exercises about sequences

This section is optional; it does not relate to our main subject.

Definition 2.1. For $a, r \in \mathbb{N}_0$ and $m \in \mathbb{N}$ we say that r is the **smallest non-negative residue of a modulo m** if $0 \leq r \leq m - 1$ and $(\exists q \in \mathbb{N}_0)(a = qm + r)$.
In this case we write $r = (a \bmod m)$.
Example: $(23 \bmod 7) = 2$ because $23 = 3 \cdot 7 + 2$.

Exercise 2.2 (Division Theorem)**.** Prove: For every $a \in \mathbb{N}_0$ and $m \in \mathbb{N}$ there is a unique $r \in \mathbb{N}_0$ such that $r = (a \bmod m)$.

Definition 2.3. The sequence $(a_n \mid n \in \mathbb{N}_0)$ is a **geometric progression** if $(\exists q)(\forall n \in \mathbb{N}_0)(a_{n+1} = qa_n)$. We call q the **quotient** of the sequence.

Definition 2.4. The sequence $(a_n \mid n \in \mathbb{N}_0)$ is a **periodic** if $(\exists d \in \mathbb{N})(\forall n \in \mathbb{N}_0)(a_{n+d} = a_n)$. In this case we call d **a period** of the sequence. We call the smallest positive period the **minimal period** of the sequence.

Example 2.5. The sequence $A, A, B, A, A, B, A, A, B, \ldots$ is periodic. Its minimal period is 3, and its periods are the positive multiples of the number 3 (i. e., $3, 6, 9, \ldots$).

Exercise 2.6. Let S be a periodic sequence with minimal period d. Prove: the periods of S are precisely the positive multiples of d, i. e., the numbers dk for $k \in \mathbb{N}$. *Hint.* Use the Division Theorem.

Exercise 2.7. (a) Find all possible minimal periods of periodic sequences of geometric progressions of real numbers.
(b) Find all possible minimal periods of periodic sequences of geometric progressions of complex numbers.

Definition 2.8. A sequence $(a_n \mid n \in \mathbb{N}_0)$ is **eventually periodic** (or **ultimately periodic**) if $(\exists d \in \mathbb{N})(\exists n_0 \in \mathbb{N}_0)(\forall n \in \mathbb{N}_0)(n \geq n_0 \Rightarrow a_{n+d} = a_n)$.

Note that being eventually periodic is a tail predicate.

Example 2.9. Consider the sequence of digits of the number
$43/84 = 0.51\ 190476\ 190476\ldots$. Show that this sequence is eventually periodic with period 6. The repeating part is 190476; and 51 is the **preperiodic part** of the sequence.
We write $43/84 = 0.51\dot{1}9047\dot{6}$ to express this circumstance; the two dots indicate the beginning and the end of the repeating part. Another example: $1/6 = 0.1\dot{6} = 0.1666\ldots$. Here the minimal period is 1.

Exercise 2.10. Let $0 < x < 1$ be a real number. Prove: (a) The sequece of digits of x is eventually periodic if and only if x is a rational number. (b) When is the sequence of digits of x (after the decimal point) periodic?

Definition 2.11. The sequence of **Fibonacci numbers** is defined by the recurrence $F_{n+2} = F_{n+1} + F_n$ $(n \in \mathbb{N}_0)$ and the initial values $F_0 = 0$, $F_1 = 1$.

So the first few terms of the Fibonacci sequence are
$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$.

Exercise 2.12. Let $m \in \mathbb{N}$. Prove that the sequence $(F_n \bmod m)$ is periodic with period $\leq m^2 - 1$.
Example: the $(F_n \bmod 3)$ sequence is periodic with period 8; the repeating part is $0, 1, 1, 2, 0, 2, 2, 1$.

Definition 2.13 (Golden ratio)**.** Let $|AB|$ denote the length of the line segment AB where A, B are points on a line. Let A, B be distinct points on a line. Consider a segment AB and the (unique) point C on the segment such that

$$\frac{|AC|}{|CB|} = \frac{|AB|}{|AC|}.$$

This proportion is called the **golden ratio** and is often denoted ϕ (the Greek letter phi).

Exercise 2.14. Prove:

$$\phi = \frac{1 + \sqrt{5}}{2} \approx 1.618 \tag{1}$$

Exercise 2.15. Prove: the ratio of a diagonal to a side of the regular pentagon is the golden ratio.

Definition 2.16. We say that the sequence $(a_n \mid n \in \mathbb{N}_0)$ is a **Fibonacci-type sequence** if for all $n \in \mathbb{N}_0$ we have $a_{n+2} = a_{n+1} + a_n$.

Exercise 2.17. Find the quotients of all nonzero Fibonacci-type geometric progressions. (A sequence is *nonzero* if at least one of its terms is nonzero.) *Hint.* There are two such ratios; one of them is the golden ratio.

Exercise 2.18 (Explicit formula for Fibonacci-type sequences)**.** If we denote the two solutions to the previous exercise by ϕ and $\overline{\phi}$, it follows that there are exactly two Fibonacci-type geometric progressions starting with 1: (ϕ^n) and $(\overline{\phi}^n)$.
Prove: every Fibonacci-type sequence is a linear combination of these two Fibonacci-type geometric progressions. In other words, if $(a_n \mid n \in \mathbb{N}_0)$ is a Fibonacci-type sequence then

$$(\exists \alpha, \beta)(\forall n)(a_n = \alpha \cdot \phi^n + \beta \cdot \overline{\phi}^n)\,. \qquad (2)$$

Exercise 2.19 (Explicit formula for the Fibonacci sequence)**.** Determine α and β in the previous exercise in the case that $a_n = F_n$ (the n-th Fibonacci number).

3 Limits

Convention 3.1. Unless expressly stated otherwise, by *sequences* we shall always mean infinite sequences of **real numbers**, possibly including a finite number of undefined terms.

We begin with reviewing the concept of a limit.

Definition 3.2 (Finite limit)**.** Let I be an infinite subset of $\mathbb{N}_0$ and $(a_n \mid n \in I)$ a sequence of real numbers. Let L be a real number. The statement *"the limit of a_n is L as n goes to infinity,"* denoted $\lim_{n\to\infty} a_n = L$ and also denoted $a_n \to L$, means that

$$(\forall \epsilon > 0)(\exists n_0)(\forall n > n_0)(|a_n - L| < \epsilon). \qquad (3)$$

In other words, for all positive values ϵ, we have $|a_n - L| < \epsilon$ for all sufficiently large values of n. Yet in other words, for all positive values ϵ, the members of the sequence will *eventually* be within an additive ϵ of L (i. e., within the interval $(L-\epsilon, L+\epsilon)$). Note that the threshold depends on ϵ.

Remark 3.3. Note that the limit defined in Eq. (3) is not affected if we change a finite number of terms in the sequence. The definition is not affected even if a finite number of terms is *undefined*. So the statement that "the limit of the sequence is a given number L" is a **tail predicate**.

Terminology 3.4. There are several other verbal expressions of the circumstance that $\lim_{n\to\infty} a_n = L$: "$a_n$ tends to L," "a_n approaches L," "a_n converges to L." (One may also say "a_n goes to L," but for finite limits L, the expression "a_n tends to L" is preferred. We usually omit the statement "as n goes to infinity" which is implicit when we speak about limits of sequences.

Exercise 3.5. Equation (3) defines *finite* limits (L is a number). Infinity is NOT a number. Give a definition, analogous to Eq. (3), that defines the statement that (a) $\lim_{n\to\infty} a_n = \infty$ (b) $\lim_{n\to\infty} a_n = -\infty$.

Terminology 3.6. We speak of a *finite limit* if L (the limit) is a number, and of an *infinite limit* if $L = \infty$ or $L = -\infty$.

WARNING 3.7. Not every sequence has a limit. It is a frequent mistake to begin reasoning about the limit of a sequence without establishing (or explicitly assuming) the existence of the limit.

Notation 3.8. $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$.

Definition 3.9. We say that the sequence (a_n) is **convergent** if it tends to a finite limit. Sequences that tend to an infinite limit or have no limit at all are called **divergent**.

Note that being convergent/divergent are *tail predicates.*

Terminology 3.10 (Verbal expressions for infinite limits)**.** If $\lim_{n\to\infty} a_n = L$ where $L \in \{\pm\infty\}$ then we say that "the limit of a_n is L," "a_n goes to L," "a_n approaches L," but we do NOT say that "a_n converges to L." One can also say "a_n tends to L," but "a_n goes to L" is preferred for infinite limits.

Exercise 3.11. Prove: if a_n approaches L , where $L \in \overline{\mathbb{R}}$, then every subsequence of (a_n) also approaches L.

Definition 3.12. We say that a sequence (a_n) is **bounded** if $(\exists C, n_0)(\forall n > n_0)(|a_n| \le C)$. We say that (a_n) is **bounded from above** if $(\exists C, n_0)(\forall n > n_0)(a_n \le C)$. "Bounded from below" is defined analogously.

Note that, by definition, boundedness is a tail predicate.

Exercise 3.13. Prove: every convergent sequence is bounded.

Exercise 3.14. Find a bounded sequence that is divergent. Use Ex. 3.11 for a very simple proof that your sequence is divergent.

Exercise 3.15. Find a divergent sequence (a_n) such that the sequence (a_n^2) is convergent.

Exercise 3.16. Prove that the constant sequence $c, c, c, \ldots$ tends to c. (Here c is a number.)

Exercise 3.17. Let K, c be numbers. Prove: if $a_n \to K$ then $ca_n \to cK$.

Exercise 3.18. (a) Prove that every eventually constant sequence is convergent.
(b) Find a convergent sequence that is not eventually constant.

Exercise 3.19. Prove: if $a_n \to K$ and $b_n \to L$, where K, L are numbers (i. e., these are finite limits) then (a) $a_n + b_n \to K + L$ and $a_n b_n \to KL$.

Exercise 3.20. Prove: if $a_n \to K$ and $b_n \to L$, where K, L are numbers, and $L \neq 0$, then $a_n/b_n \to K/L$.

Exercise 3.21. (a) Prove: if $a_n \to K$ where K is a number and $a_n \ge 0$ then $\sqrt{a_n} \to \sqrt{K}$.
(b) More generally, let f be a continuous function whose domain includes the range (set of values) of the sequence (a_n) and the number L. Prove: if $a_n \to L$ then $f(a_n) \to f(L)$.
(c) Prove: if $a_n \to K$ where K is a number and $K > 0$ then $\sqrt{a_n} \to \sqrt{K}$. (Note that $\sqrt{x}$ is only defined for non-negative real numbers x.)

Definition 3.22. The sequence $(a_n \mid n \in \mathbb{N}_0)$ is **nondecreasing** if $(\forall n)(a_n \le a_{n+1})$. The sequence is **(strictly) increasing** if $(\forall n)(a_n < a_{n+1})$.

Exercise 3.23. Prove: every nondecreasing sequence has a limit.
In particular, if a nondecreasing sequence is bounded from above then it is convergent.

Fact 3.24. One of the most important limit relations is the following.

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \mathrm{e}\,. \tag{4}$$

This is part theorem, part definition. First we prove that the sequence on the left-hand side is strictly increasing and bounded, thus showing that the left-hand side converges. Then we define the number e as the value of this limit.
The following more general result holds.

Fact 3.25. Let $z \in \mathbb{R}$. Then

$$\lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n = \mathrm{e}^z\,. \tag{5}$$

Exercise 3.26. Let us define the sequence (s_n) by the recurrence $s_{n+1} = (\sqrt{2})^{s_n}$ and the initial value $s_0 = 0$. Prove:
(a) s_n is strictly increasing
(b) $(\forall n)(s_n < 2)$
(c) Note that from (a) and (b) it follows that this sequence converges
(d) $s_n \to 2$.

Exercise 3.27. Let $c > 1$ and let us define the sequence $(t_n(c))$ by the recurrence $t_{n+1}(c) = c^{t_n(c)}$ and the initial value $t_0 = 0$. Find the largest value c such that the sequence $t_n(c)$ is bounded (and therefore convergent) as $n \to \infty$.

Exercise 3.28. Let F_n denote the n-th Fibonacci number (see Def. 2.11). Give a very simple proof of the following fact.
If the quotients F_{n+1}/F_n converge then their limit is the golden ratio (see Def. 2.13).
Do not use the explicit formula for Fibonacci numbers (Exercises 2.18 and 2.19).
This is a case when it is easier to compute the limit assuming it exists, than proving the existence of the limit.

4 Upper/lower limit (limsup, liminf)

First we define the supremum of a set of real numbers.

Definition 4.1 (Upper bound)**.** Let $S \subseteq \mathbb{R}$. We say that $L \in \overline{\mathbb{R}}$ is an **upper bound** of S if $(\forall x \in S)(x \le L)$.

Exercise 4.2. Prove that $-\infty$ is an upper bound of $S \subseteq \mathbb{R}$ if and only if $S = \emptyset$.

Definition 4.3 (Supremum)**.** Let $S \subseteq \mathbb{R}$. We say that L is the **supremum** of S if (a) L is an upper bound of S, and (b) for every upper bound M of S we have $L \le M$. The supremum of S is also called the **least upper bound** of S and is denoted $\sup(S)$.

The *existence* of the supremum is a basic fact about the ordering of the real numbers.

Fact 4.4. Every subset of $\mathbb{R}$ has a least upper bound.

Exercise 4.5. Show that the least upper bound is unique (every set has only one least upper bound).

Definition 4.6. Lower bounds and least lower bound are defined analogously. The **least lower bound** is also called the **infimum** and is denoted $\inf(S)$.

Notation 4.7. For $S \subseteq \mathbb{R}$ we write $-S = \{-x \mid x \in S\}$.

Exercise 4.8. Let $S \subseteq \mathbb{R}$. Prove: $\inf(S) = -\sup(-S)$.

Exercise 4.9. For which subsets S of $\mathbb{R}$ is $\sup(S) < \inf(S)$?

Definition 4.10. A set $S \subseteq \mathbb{R}$ is **bounded from above** if it has a finite upper bound. Boudnedness from below is defined analogously.

Exercise 4.11. Prove: $S \subseteq \mathbb{R}$ is bounded from above if and only if $\sup(S) < \infty$.

Definition 4.12. A set $S \subseteq \mathbb{R}$ is **bounded** if S is bounded from above and bounded from below.

Exercise 4.13. Show that $S \subseteq \mathbb{R}$ is bounded if and only if $\{|x| : x \in S\}$ is bounded from above.

Exercise 4.14. Observe that $L \in \overline{\mathbb{R}}$ is an upper bound of the sequence $(a_n \mid n \in I)$ if and only if L is an upper bound of the set $\text{range}(S) := \{a_n \mid n \in I\}$.

Definition 4.15. The **supremum of a sequence** is the supremum of its range. We define the **infimum of a sequence** analogously. Notation: $\sup_{n\in I} a_n$ and $\inf_{n\in I} a_n$.

Definition 4.16 (Limsup)**.** The **upper limit** or **limsup** of the sequence $(a_n \mid n \in I)$ is defined as

$$\limsup_{n\in I} a_n = \lim_{i\to\infty} A_i \tag{6}$$

where, for $i \in \mathbb{N}_0$, we set $A_i = \sup\{a_n \mid n \in I, n \geq i\}$.

Exercise 4.17. Prove that the sequence (A_i) is nonincreasing. This guarantees that the limit on the right-hand side of Eq. (6) exists.

Exercise 4.18. For $L \in \overline{\mathbb{R}}$, the statement "$\limsup a_n = L$" is a tail predicate. Same with $\liminf$.

Exercise 4.19 (characterization of limsup)**.** Let $M \in \mathbb{R}$ and let (a_n) be a sequence. Prove: $\limsup a_n \leq M$ if and only if for all $\epsilon > 0$, eventually $a_n < M + \epsilon$.

Exercise 4.20 (limsup vs limit of subsequence)**.** Let $L \in \overline{\mathbb{R}}$ and let $(a_n \mid n \in I)$ be a sequence. Prove: if $\limsup_{n\in I} a_n = L$ then there exists a subsequence $(a_n \mid n \in J)$ (where $J \subseteq I$ is an infinite set) such that $\lim_{n\in J} a_n = L$.

Exercise 4.21. For every sequence S, $\liminf(S) \leq \limsup(S)$.

The great advantage of using limsup and liminf is that, in contast to the limit, these two quantities always exist.

Exercise 4.22 (existence of limit)**.** Let S be a sequence. Prove: $\lim(S)$ exists if and only if $\liminf(S) = \limsup(S)$.

Exercise 4.23. Let $a_n = (-1)^n(1 + 1/n)$. Determine $\limsup a_n$ and $\liminf a_n$.

Exercise 4.24 (subadditivity and submultiplicativity of limsup)**.**

(a) Let (a_n) and (b_n) be sequences. Prove: $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n)$, assuming the right-hand side makes sense. (The only case when it does not make sense is when one of the terms is ∞ and the other is $-\infty$.)

(b) Let (a_n) and (b_n) be sequences of non-negative numbers. Prove: $\limsup(a_n b_n) \leq (\limsup a_n) \cdot (\limsup b_n)$, assuming the right-hand side makes sense. (The only case when it does not make sense is when one of the terms is 0 and the other is ∞.)

Exercise 4.25. Prove: If $\limsup a_n = -\infty$ then $\lim a_n = -\infty$.

Exercise 4.26.

(a) Find two bounded sequences, (a_n) and (b_n), such that $\limsup(a_n + b_n) < \limsup a_n + \limsup b_n$.

(b) Find two sequences, (a_n) and (b_n), such that $\limsup(a_n + b_n) = -\infty$ while $\limsup a_n = \limsup b_n = \infty$.

5 Standard definition and examples of asymptotic equality

Often, we are interested in comparing the rate of growth of two functions, as inputs increase in length. Asymptotic equality is one formalization of the idea of two functions having the "same rate of growth."

Convention 5.1. Henceforth, the domains of our sequences will be $\mathbb{N}$ or $\mathbb{N}_0$. So we shall write $\limsup_{n\to\infty} a_n$ instead of $\limsup_{n\in I} a_n$, and analogously for lim inf and lim. But we understand that all definitions and results easily extend to sequences of which the domain is a subset of $\mathbb{N}_0$.

Let (a_n) and (b_n) be two sequences.

Definition 5.2 (Standard definition of asymptotic equality)**.** We say that a_n is *asymptotically equal* to b_n if $\lim_{n\to\infty} a_n/b_n = 1$. We denote this circumstance by $a_n \sim b_n$.

Exercise 5.3. Prove: if $a_n \sim b_n$ and $c_n \sim d_n$ then $a_n c_n \sim b_n d_n$ and $a_n/c_n \sim b_n/d_n$. (Recall that a finite number of undefined terms do not invalidate a limit relation.)

WARNING 5.4. Asymptotic equality does not mean eventual equality, as, for instance, example (a) in the next exercise shows.

Exercise 5.5. (a) Prove: $n^2 + n \sim n^2$

(b) Prove: $5n^3 + 7n - 1000 \sim 5n^3$

(c) Prove: $\dfrac{5n^3 + 7n - 1000}{8n^{10} - 6n^2 - 9n + 7} \sim \dfrac{5}{8n^7}$

(d) Note that for $n = 1$, the denominator in the preceding example is zero. Explain why this is not a problem.

(e) Prove: there exist real numbers a, b such that $\binom{n}{5} \sim a \cdot n^b$.

Find a, b. Make your proof elegant.

Definition 5.6. A **polynomial** f is a function defined by an expression of the form

$$f(x) = \sum_{j=0}^{n} a_k x^k \tag{7}$$

where the **coefficients** a_k are (real) numbers. The **degree** of f, denoted $\deg(f)$, is the largest k such that $a_k \neq 0$. This a_k is called the *leading coefficient* of f and $a_k x^k$ is the *leading term* of f. If all coefficients of f are zero then f is the *zero polynomial*, which we denote by 0, and by convention, $\deg(0) = -\infty$.

Exercise 5.7. If f and g are polynomials then $\deg(fg) = \deg(f) + \deg(g)$ and $\deg(f+g) \leq \max\{\deg(f), \deg(g)\}$.

Exercise 5.8. Prove:

(a) If $f(x)$ and $g(x)$ are polynomials with respective leading terms ax^k and bx^ℓ then $f(n)/g(n) \sim (a/b)n^{k-\ell}$.

(b) $\sin(1/n) \sim 1/n$.

(c) $\ln(1+1/n) \sim 1/n$.

(d) $\sqrt{n^2+1} - n \sim 1/(2n)$.

Exercise 5.9. Prove:

(a) If f is a function, differentiable at zero, $f(0) = 0$, and $f'(0) \neq 0$, then $f(1/n) \sim f'(0)/n$.

(b) Show that items (b), (c), (d) of the preceding exercise follow from part (a) of this exercise.

Exercise 5.10. Find two sequences of positive real numbers, (a_n) and (b_n), such that $a_n \sim b_n$ but $a_n^n \not\sim b_n^n$.

Next we state some of the most important asymptotic relations in mathematics.

Theorem 5.11 (Stirling's Formula)**.**

$$n! \sim \left(\frac{n}{\mathrm{e}}\right)^n \sqrt{2\pi n}.$$

Exercise 5.12. Prove: there exist real numbers a, b, c such that $\binom{2n}{n} \sim a \cdot n^b \cdot c^n$. Find a, b, c.

Next we state one of the most beautiful theorems of all of mathematics.

Definition 5.13 (Prime counting function)**.** Let $\pi(x)$ denote the number of primes less than or equal to x.

So $\pi(10) = 4$, $\pi(100) = 25$, $\pi(\pi) = 2$, $\pi(-10) = 0$.

Theorem 5.14 (The Prime Number Theorem)**.**

$$\pi(x) \sim \frac{x}{\ln x},$$

where ln *denotes the natural logarithm function.*

This fundamental result was proved in 1896 independently by Jacques Hadamard and Charles de la Vallée Poussin, using the theory of complex functions (Riemann's zeta function).

Exercise 5.15. *Feasibility of generating random prime numbers.* Estimate, how many random $\leq$ 100-digit integers should we expect to pick before we encounter a prime number. (We generate our numbers by choosing the 100 digits independently at random (initial zeros are permitted), so each of the 10^{100} numbers has the same probability to be chosen.) Interpret this question as asking the reciprocal of the probability that a randomly chosen integer is prime. Assume, for the purposes of this estimate, that 10^{100} is sufficiently large for the Prime Number Theorem to give a good estimate. State how good the estimate needs to be.

6 Properties of asymptotic equality

The following simple exercise is conceptually significant.

Exercise 6.1. If $a_n \sim b_n$ then both of these sequences are eventually nonzero.

Let c be a (real or complex) number. We write $\underline{c}$ to denote the sequence $(c, c, \dots)$.

Exercise 6.2. If $c \neq 0$ is a number then the statement $a_n \sim \underline{c}$ is equivalent to $a_n \to c$.

Note that this is false for $c = 0$: the sequence $\underline{0}$ is not asymptotically equal to any sequence, not even to itself, according to Ex. 6.1.

Exercise 6.3. Let $\mathcal{S}$ denote the set of eventually nonzero sequences. Prove that $\sim$ is an *equivalence relation* on $\mathcal{S}$, i. e., the relation "$\sim$" is

(a) *reflexive:* $a_n \sim a_n$;

(b) *symmetric:* if $a_n \sim b_n$ then $b_n \sim a_n$; and

(c) *transitive:* if $a_n \sim b_n$ and $b_n \sim c_n$ then $a_n \sim c_n$.

(Implicit universal quantifiers: The statements above hold for all sequences (a_n), (b_n), $(c_n) \in \mathcal{S}$.)

Exercise 6.4. Consider the following statement.

$$\text{If } a_n \sim b_n \text{ and } c_n \sim d_n \text{ then } a_n + c_n \sim b_n + d_n. \tag{8}$$

(a) Prove that (8) is false, even if we assume that all the six sequences involved are eventually nonzero.

(b) Prove: if $a_n c_n > 0$ for all sufficiently large n then (8) is true.
Hint. Prove: if $a, b, c, d > 0$ and $a/b < c/d$ then $a/b < (a + c)/(b + d) < c/d$.

We say that the "statement A *implies* statement B" if B follows from A.

Exercise 6.5. Assume $a_n, b_n > 1$. Consider the following statements:

(A) $a_n \sim b_n$;

(B) $\ln a_n \sim \ln b_n$.

Prove:

(a) (A) does not imply (B).

(b) (A) does imply (B) under the stronger assumption that $a_n \geq 1.01$.

Definition 6.6. We say that the sequence a_n is *bounded away from the number L* if there exists $c > 0$ such that for all sufficiently large n we have $|a_n - L| > c$.

The condition $a_n \geq 1.01$ in the preceding exercise can be replaced by the condition that a_n is bounded away from 1 (while we continue to assume $a_n > 1$).

Exercise 6.7 (Squeeze principle)**.** Assume that for all sufficiently large n we have $a_n \leq b_n \leq c_n$. Assume further that $a_n \sim c_n$. Prove: $a_n \sim b_n \sim c_n$.

Exercise 6.8. (a) Prove: for all $n \geq 1$ we have $n! > \left(\frac{n}{e}\right)^n$.
Use the power series expansion of e^x.
(b) Reason, why this result does not follow from Stirling's formula.
(c) Show that this result does follow from Stirling's formula for all sufficiently large n.

Exercise 6.9. Prove: $\ln(n!) \sim n \ln n$. Give two proofs. (a) Use Stirling's formula. (b) Do not use Stirling's formula; use Ex. 6.8 instead.

Exercise 6.10. Let p_n be the n-th prime number. Consider the following asymptotic equality: $p_n \sim n \ln n$. Prove that this statement is equivalent to the Prime Number Theorem.

Exercise$^+$ 6.11. Let $P(x)$ denote the product of all prime numbers $\leq x$. Consider the following statement: $\ln P(x) \sim x$. Prove that this statement is equivalent to the Prime Number Theorem.

7 Extended definition of asymptotic equality

We wish the relation of asymptotic equality to be an equivalence relation among *all sequences*. This is not true under the standard definition; the relation is not even reflexive: the all-zero sequence $\underline{0}$ is not asymptotically equal to itself.

Exercise 7.1. $a_n \sim a_n$ if and only if the sequence (a_n) is eventually nonzero.

To remedy this, we shall replace every occurrence of the fraction $0/0$ by 1.

Remark 7.2. This does not mean that we think $0/0 = 1$. No, the fraction $0/0$ continues to be undefined. This is simply a technical trick, we *replace* all occurrences of the fraction $0/0$ among the fractions a_n/b_n by 1. The benefits of this trick will be apparent below. I note that this is not a standard trick, you will not find it in any textbook.

So the exact definition goes as follows.

Definition 7.3 (Extended definition of asymptotic equality)**.** Let (a_n) and (b_n) be two sequences. Let $q_n = a_n/b_n$ except when $a_n = b_n = 0$; in that case, let $q_n = 1$. We say that the sequence (a_n) is *asymptotically equal* to the sequence (b_n) *under the extended definition* if $\lim_{n\to\infty} q_n = 1$. We denote this circumstance by $a_n \overset{*}{\sim} b_n$.

Note that if $a_n \neq 0$ but $b_n = 0$ then q_n is undefined; and if infinitely many terms of the sequence $\{q_n\}$ are undefined then the sequence has no limit.

Exercise 7.4. Our extended definition indeed *extends* the standard definition:

(a) If $a_n \sim b_n$ then $a_n \overset{*}{\sim} b_n$.

(b) If at least one of the sequences (a_n) and (b_n) is eventually nonzero, then $a_n \sim b_n$ if and only if $a_n \overset{*}{\sim} b_n$.

The converse of item (a) is not true: $\underline{0} \overset{*}{\sim} \underline{0}$ but $\underline{0} \not\sim \underline{0}$.
Compare item (b) with Ex. 6.1.

8 Properties of extended asymptotic equality

From now on we use our extended definition of asymptotic equality, Def. 7.3. Remember, however, that if we restrict our attention to *eventually nonzero* sequences then there is no difference between the two concepts.

Exercise 8.1. Prove: $a_n \overset{*}{\sim} \underline{0}$ if and only if a_n is eventually zero.

Exercise 8.2. Let $\mathcal{S}^*$ denote the set of all sequences of real or complex numbers. Prove that $\overset{*}{\sim}$ is an *equivalence relation* on $\mathcal{S}^*$, i. e., the relation "$\overset{*}{\sim}$" is

(a) *reflexive:* $a_n \overset{*}{\sim} a_n$;

(b) *symmetric:* if $a_n \overset{*}{\sim} b_n$ then $b_n \overset{*}{\sim} a_n$; and

(c) *transitive:* if $a_n \overset{*}{\sim} b_n$ and $b_n \overset{*}{\sim} c_n$ then $a_n \overset{*}{\sim} c_n$.

(Implicit universal quantifiers: The statements above hold for all sequences (a_n), (b_n), (c_n).)

Exercise 8.3. Prove: if $a_n \overset{*}{\sim} b_n$ and $c_n \overset{*}{\sim} d_n$ then $a_n c_n \overset{*}{\sim} b_n d_n$. If, moreover, $c_n d_n \neq 0$ for all sufficiently large n then $a_n/c_n \overset{*}{\sim} b_n/d_n$.

9 Asymptotic inequality

We wish to define a notion of asymptotic inequality between sequences, to be called the "greater than or asymptotically equal" relation, denoted $a_n \gtrsim b_n$. It is natural to expect our definition to satisfy the following conditions:

(a) $a_n \overset{*}{\sim} b_n$ if and only if both $a_n \gtrsim b_n$ and $b_n \gtrsim a_n$ hold;

(b) if $(\forall n)(a_n \geq b_n)$ then $a_n \gtrsim b_n$; and

(c) if $a_n \gtrsim b_n$ and $a_n \gtrsim c_n$ then $a_n \gtrsim \max\{b_n, c_n\}$.

It turns out that these three conditions together already determine the concept. The simplest definition we could find that works for all pairs of sequences is described next. A more intuitive definition that works in the important case of sequences of positive numbers will be given in Exercise 9.15 below. The definition will be followed by a series of exercises each of which can be solved in a few lines given the exercises preceding it; these exercises reveal the basic properties of the $\gtrsim$ relation.

Definition 9.1. Let (a_n) and (b_n) be sequences of real numbers. We say that a_n is *greater than or asymptotically equal to* b_n, denoted as $a_n \gtrsim b_n$ if

$$a_n \overset{*}{\sim} \max\{a_n, b_n\} . \tag{9}$$

A dual definition, using min, can also be given; the two definitions are equivalent (Exercise 9.6). Before proving the equivalence, we verify some simple consequences of the definition given.

Exercise 9.2. Prove: if $a_n \overset{*}{\sim} b_n$ then $a_n \gtrsim b_n$.

Exercise 9.3. Prove: if $a_n \geq b_n$ holds for all sufficiently large n then $a_n \gtrsim b_n$.

Exercise 9.4. (a) Prove: if $a_n \gtrsim b_n$ and $b_n \gtrsim a_n$ then $a_n \stackrel{*}{\sim} b_n$.

(b) [Squeeze principle] Prove: if $a_n \lesssim b_n \lesssim c_n$ and $a_n \stackrel{*}{\sim} c_n$ then $a_n \stackrel{*}{\sim} b_n \stackrel{*}{\sim} c_n$.

(c) Prove: if $a_n \lesssim b_n \lesssim c_n$ and $a_n \sim c_n$ then $a_n \sim b_n \sim c_n$.

These facts are immediate from the definition. Now we give a somewhat technical yet intuitive equivalent definition.

Exercise 9.5. Let (a_n) and (b_n) be sequences of real numbers. Let $B = \{n : a_n < b_n\}$. We claim that $a_n \gtrsim b_n$ if and only if either B is finite or the subsequences $(a_n : n \in B)$ and $(b_n : n \in B)$ are asymptotically equal (in the extended sense).

This exercise seems to justify the term "greater than or asymptotically equal:" for some of the subscripts, $a_n \geq b_n$; and for the remaining subscripts, $a_n \stackrel{*}{\sim} b_n$.
The dual characterization of the $\gtrsim$ relation is now immediate.

Exercise 9.6. Prove: $a_n \gtrsim b_n$ if and only if $b_n \stackrel{*}{\sim} \min\{a_n, b_n\}$.

The following is an immediate corollary:

Exercise 9.7. $a_n \gtrsim b_n$ if and only if $-b_n \gtrsim -a_n$.

We begin our preparations for proving that the $\gtrsim$ relation is transitive. The following two exercises give the two most useful and intuitive characterizations of asymptotic inequality.

Exercise 9.8 (Characterization of asymptotic inequality 1)**.** Prove: $a_n \gtrsim b_n$ if and only if there exists a sequence d_n such that $a_n \stackrel{*}{\sim} d_n \geq b_n$.

Note that the $\Rightarrow$ direction is immediate from the definition; the $\Leftarrow$ direction follows using Exercise 9.5.

Exercise 9.9 (Characterization of asymptotic inequality 2)**.** Prove: $a_n \gtrsim b_n$ if and only if there exists a sequence c_n such that $a_n \geq c_n \stackrel{*}{\sim} b_n$.
Hint. This exercise is an immediate consequence of the preceding exercise by switching signs and using Exercise 9.7.

Now we are ready to prove transitivity.

Exercise 9.10. Prove: if $a_n \gtrsim b_n$ and $b_n \gtrsim c_n$ then $a_n \gtrsim c_n$.

The proof only requires some manipulation of symbols using the preceding two exercises. Indeed, notice that there exist sequences $\{u_n\}$ and $\{v_n\}$ such that $a_n \geq u_n \stackrel{*}{\sim} b_n \stackrel{*}{\sim} v_n \geq c_n$. It follows that $a_n \gtrsim v_n$ and therefore there exists a sequence $\{w_n\}$ such that $a_n \stackrel{*}{\sim} w_n \geq v_n$; the conclusion $a_n \gtrsim c_n$ is now immediate.

Exercise 9.11. Conclude from the preceding exercises that the "$\gtrsim$" relation is a partial order on the set of extended asymptotic equivalence classes of sequences of real numbers.

Exercise 9.12. Prove: $a_n \gtrsim \underline{0}$ if an only if a_n is eventually non-negative, i. e., $a_n \geq 0$ for all sufficiently large n.

Hint: Exercises 9.9 and 8.1.

Exercise 9.13. True or false:
(a) If $a_n \gtrsim b_n$ and b_n is eventually non-negative then a_n must also eventually be non-negative.
(b) If $a_n \gtrsim b_n$ and b_n is eventually positive then a_n must also eventually be positive.

Exercise 9.14. Prove: If $a_n \gtrsim b_n$ and $c_n \gtrsim d_n$ and (b_n) and (d_n) are eventually non-negative then $a_n c_n \gtrsim b_n d_n$.

Next we turn to sequences of positive numbers; in this case, the following more intuitive characterization of the $\gtrsim$ relation be given.

Exercise 9.15 (asymptotic inequality for sequences of positive numbers)**.** Let (a_n) and (b_n) be sequences of positive numbers. Prove: $a_n \gtrsim b_n$ if and only if $\liminf a_n/b_n \geq 1$.

An equivalent version of this characterization is the following:

Exercise 9.16. Let $a_n, b_n \geq 0$. Prove: $a_n \gtrsim b_n$ if and only if for all $\epsilon > 0$, we have $a_n \geq (1-\epsilon)b_n$ for all sufficiently large n.

(As usual, the threshold depends on ϵ.)

Exercise 9.17. Show that if a_n and b_n are sequences of negative numbers, then the condition $\liminf a_n/b_n \geq 1$ is neither necessary, nor sufficient for the relation $a_n \gtrsim b_n$.

However, we do have a characterisation of the $\gtrsim$ relation, analogous to Ex. 9.15, for sequences are negative numbers.

Exercise 9.18 (asymptotic inequality for sequences of negative numbers)**.** Let (a_n) and (b_n) be sequences of negative numbers. Prove: $a_n \gtrsim b_n$ if and only if $\limsup a_n/b_n \leq 1$.

Exercise 9.19. Let $a_n, b_n \leq 0$. Prove: $a_n \gtrsim b_n$ if and only if for all $\epsilon > 0$, we have $a_n \leq (1+\epsilon)b_n$ for all sufficiently large n.

We now wish to characterize asymptotic inequality for arbitrary sequences of real numbers.

Exercise 9.20. Let (a_n) and (b_n) be arbitrary sequences of real numbers. Let $P = \{n \mid a_n \geq 0$ and $b_n > 0\}$,
$N = \{n \mid a_n \leq 0 \text{ and } b_n < 0\}$,
$U = \{n \mid a_n \geq 0 \text{ and } b_n \leq 0\}$,
and $D = \{n \mid a_n \leq 0 \text{ and } b_n \geq 0\}$.
Prove: $a_n \gtrsim b_n$ if and only if all of the following hold:
(i) either P is finite or $\liminf_{n\in P} a_n/b_n \geq 1$
(ii) either N is finite or $\limsup_{n\in N} a_n/b_n \leq 1$
(iii) either D is finite or both $(a_n \mid n \in D)$ and $(b_n \mid n \in D)$ are eventually zero.

Exercise 9.21. (a) Prove: If $a_n - b_n \gtrsim \underline{0}$ then $a_n \gtrsim b_n$.

(b) The converse is false: $a_n \gtrsim b_n$ does not imply $a_n - b_n \gtrsim \underline{0}$.

Exercise 9.22. (a) Prove: If $a_n \gtrsim b_n$ and $c_n \gtrsim d_n$ and (b_n) and (d_n) are eventually non-negative then $a_n + c_n \gtrsim b_n + d_n$.

(b) Show that the conclusion would not follow without the non-negativity assumption.

Exercise 9.23. Assume $a_n, b_n > 1$ for all sufficiently large n. Consider the following statements:

(A) $a_n \gtrsim b_n$;

(B) $\ln a_n \gtrsim \ln b_n$.

Prove:

(a) (A) does not imply (B).

(b) (A) does imply (B) under the stronger assumption that $(\exists c > 0)(\exists n_0)(\forall n \geq n_0)(a_n \geq 1 + c)$ (in other words, a_n is bounded away from 1 in the positive direction). (See Exercise 6.5.)

Exercise$^+$ 9.24. Let $a_n \geq 1$. Prove: $a_n^2 \ln a_n \gtrsim n$ if and only if $a_n \gtrsim \sqrt{2n/\ln n}$.

Exercise 9.25. We are given n distinct weights and want to sort them using a balance to compare pairs of weights. The weights are given in any order, so there are $n!$ possible inputs.

(a) Suppose an algorithm sorts every input using $\leq C(n)$ comparisons. Prove: $C(n) \gtrsim n \log_2 n$.

(b) Prove the same conclusion if the algorithm is only required to work correctly one percent of the time, i. e., it will correctly sort $n!/100$ inputs.

Exercise 9.26. (a) Find a sequence (a_n) of positive numbers such that $a_{n+1} \gtrsim a_n$ and $a_n \to 0$.

(b) Let (a_n) be a sequence of positive numbers such that $a_{n+1} \gtrsim a_n$. Show that a_n cannot decrease exponentially, i. e., for all $0 < c < 1$, we have $a_n > c^n$ for all sufficiently large n.