Asymptotic Equality and Inequality

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1 Asymptotic equality

Often, we are interested in comparing the rate of growth of two functions, as inputs increase in length. Asymptotic equality is one formalization of the idea of two functions having the "same rate of growth."

Let $\{a_n\}$ and $\{b_n\}$ be two sequences.

We'd like to say that a_n is *asymptotically equal* to b_n (denoted $a_n \sim b_n$) if $\lim_{n\to\infty} a_n/b_n = 1$.

We also wish the relation of asymptotic equality to be an equivalence relation on sequences. As stated, however, the definition is not even reflexive: the all-zero sequence would not be asymptotically equal to itself (because the fraction $0/0$ is undefined).

To remedy this, we shall replace every occurrence of the fraction $0/0$ by 1. So the exact definition goes as follows.

Definition 1.1. Let $\{a_n\}$ and $\{b_n\}$ be two sequences. Let $q_n = a_n/b_n$ except when $a_n = b_n = 0$; in that case, let $q_n = 1$. We say that the sequence $\{a_n\}$ is *asymptotically equal* to the sequence $\{b_n\}$ if $\lim_{n\to\infty} q_n = 1$.

Note that if $a_n \neq 0$ but $b_n = 0$ then q_n is undefined; and if infinitely many terms of the sequence $\{q_n\}$ are undefined then the sequence has no limit.

Observation. If $c \neq 0$ is a constant then the statement $a_n \sim c$ (where c means the sequence $c, c, \dots$) is equivalent to $a_n \to c$ (where c means the number c).

Exercise 1.2. Prove: $a_n \sim 0$ if and only if $(\exists n_0)(\forall n \geq n_0)(a_n = 0)$, i.e., $a_n = 0$ for all sufficiently large n.

Exercise 1.3. Let $\mathcal{S}$ denote the set of sequences of real or complex numbers. Prove that $\sim$ is an *equivalence relation* on $\mathcal{S}$, i.e., the relation "$\sim$" is

(a) *reflexive:* $a_n \sim a_n$;

(b) *symmetric:* if $a_n \sim b_n$ then $b_n \sim a_n$; and

(c) *transitive:* if $a_n \sim b_n$ and $b_n \sim c_n$ then $a_n \sim c_n$.

Exercise 1.4. Prove: if $a_n \sim b_n$ and $c_n \sim d_n$ then $a_n c_n \sim b_n d_n$. If, moreover, $c_n d_n \neq 0$ for all sufficiently large n then $a_n/c_n \sim b_n/d_n$. (Note that a finite number of undefined terms do not invalidate a limit relation.)

Exercise 1.5. Consider the following statement.

$$\text{If } a_n \sim b_n \text{ and } c_n \sim d_n \text{ then } a_n + c_n \sim b_n + d_n. \tag{1}$$

1. Prove that (1) is false.
2. Prove: if $a_n c_n > 0$ then (1) is true. *Hint.* Prove: if $a, b, c, d > 0$ and $a/b < c/d$ then $a/b < (a+c)/(b+d) < c/d$.

Exercise 1.6.

1. If $f(x)$ and $g(x)$ are polynomials with respective leading terms ax^n and bx^m then $f(n)/g(n) \sim (a/b)x^{n-m}$.
2. $\sin(1/n) \sim \ln(1 + 1/n) \sim 1/n$.
3. $\sqrt{n^2+1} - n \sim 1/2n$.
4. If f is a function, differentiable at zero, $f(0) = 0$, and $f'(0) \neq 0$, then $f(1/n) \sim f'(0)/n$. See that items 2 and 3 in this exercise follow from this.

Exercise 1.7. Find two sequences of positive real numbers, $\{a_n\}$ and $\{b_n\}$, such that $a_n \sim b_n$ but $a_n^n \not\sim b_n^n$.

Next we state some of the most important asymptotic formulas in mathematics.

Theorem 1.8 (Stirling's Formula)**.**

$$n! \sim \left(\frac{n}{\mathrm{e}}\right)^n \sqrt{2\pi n}.$$

Exercise 1.9. Prove: $\displaystyle\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}$.

We say that the "statement A *implies* statement B" if B follows from A.

Exercise 1.10. Assume $a_n, b_n > 1$. Consider the following statements:

(A) $a_n \sim b_n$;

(B) $\ln a_n \sim \ln b_n$.

Prove:

(a) (A) does not imply (B).

(b) (A) does imply (B) under the stronger assumption that $a_n \geq 1.01$.

We say that the sequence $a_n \geq 1$ is *bounded away from 1* if $(\exists c > 0)(\exists n_0)(\forall n \geq n_0)(a_n \geq 1 + c)$. The condition $a_n \geq 1.01$ in the preceding exercise can be replaced by the condition that a_n is bounded away from 1.

Exercise 1.11. Prove: $\ln(n!) \sim n \ln n$. Give two proofs. (1) In the first proof, use Stirling's formula and Exercise 1.10 (b). (2) The second proof should be from first principles, from the definition of $n!$ and the definition of asymptotic equality.

Theorem 1.12 (The Prime Number Theorem). *Let $\pi(x)$ be the number of primes less than or equal to x.*

$$\pi(x) \sim \frac{x}{\ln x},$$

where ln *denotes the natural logarithm function.*

Exercise 1.13. Let p_n be the n-th prime number. Prove, using the Prime Number Theorem, that $p_n \sim n \ln n$.

Exercise 1.14. *Feasibility of generating random prime numbers.* Estimate, how many random ≤ 100-digit integers should we expect to pick before we encounter a prime number? (We generate our numbers by choosing the 100 digits independently at random (initial zeros are permitted), so each of the 10^{100} numbers has the same probability to be chosen.) Interpret this question as asking the reciprocal of the probability that a randomly chosen integer is prime.

Exercise$^+$ 1.15. Let $P(x)$ denote the product of all prime numbers $\leq x$. Consider the following statement: $\ln P(x) \sim x$. Prove that this statement is equivalent to the Prime Number Theorem.

2 Asymptotic inequality

We wish to define a notion of asymptotic inequality between sequences, to be denoted $a_n \gtrsim b_n$. It is natural to expect our definition to satisfy the following conditions:

(a) $a_n \sim b_n$ if and only if both $a_n \gtrsim b_n$ and $b_n \gtrsim a_n$ hold;

(b) if $a_n \geq b_n$ then $a_n \gtrsim b_n$; and

(c) the relation $\gtrsim$ is transitive.

It turns out that these three conditions together already determine the concept. The simplest definition we could find that works for all pairs of sequences is described next. A perhaps more intuitive definition that works in the important case of sequences of positive numbers will be given in Exercise *** below. The definition will be followed by a series of exercises each of which can be solved in a few lines given the exercises preceding it; these exercises reveal the basic properties of the $\gtrsim$ relation.

Definition 2.1. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. We say that a_n is *greater than or asymptotically equal to* b_n, denoted as $a_n \gtrsim b_n$ if $a_n \sim \max\{a_n, b_n\}$.

A dual definition, using min, can also be given; the two definitions are equivalent (Exercise 2.6). Before proving the equivalence, we verify some simple consequences of the definition given.

Exercise 2.2. Prove: if $a_n \sim b_n$ then $a_n \gtrsim b_n$.

Exercise 2.3. Prove: if $a_n \geq b_n$ then $a_n \gtrsim b_n$.

Exercise 2.4. Prove: if $a_n \gtrsim b_n$ and $b_n \gtrsim a_n$ then $a_n \sim b_n$.

These facts are immediate from the definition. Now we give a somewhat technical yet intuitive equivalent definition.

Exercise 2.5. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. Let $B = \{n : a_n < b_n\}$. We claim that $a_n \gtrsim b_n$ if and only if either B is finite or the subsequences $\{a_n : n \in B\}$ and $\{b_n : n \in B\}$ are asymptotically equal.

This exercise seems to justify the term "greater than or asymptotically equal:" for some of the subscripts, $a_n \geq b_n$; and for the remaining subscripts, $a_n \sim b_n$.

The dual characterization of the $\gtrsim$ relation is now immediate.

Exercise 2.6. Prove: $a_n \gtrsim b_n$ if and only if $b_n \sim \min\{a_n, b_n\}$.

The following is an immediate corollary:

Exercise 2.7. $a_n \gtrsim b_n$ if and only if $-b_n \gtrsim -a_n$.

We begin our preparations for proving that the $\gtrsim$ relation is transitive.

Exercise 2.8. Prove: $a_n \gtrsim b_n$ if and only if there exists a sequence d_n such that $a_n \sim d_n \geq b_n$.

Note that the $\Rightarrow$ direction is immediate from the definition; the $\Leftarrow$ direction follows using Exercise 2.5.

Exercise 2.9. Prove: $a_n \gtrsim b_n$ if and only if there exists a sequence c_n such that $a_n \geq c_n \sim b_n$.

This exercise is an immediate consequence of the preceding exercise by switching signs and using Exercise 2.7.

Now we are ready to prove transitivity.

Exercise 2.10. Prove: if $a_n \gtrsim b_n$ and $b_n \gtrsim c_n$ then $a_n \gtrsim c_n$.

The proof only requires some manipulation of symbols using the preceding two exercises. Indeed, Notice that there exist sequences $\{u_n\}$ and $\{v_n\}$ such that $a_n \geq u_n \sim b_n \sim v_n \geq c_n$. It follows that $a_n \gtrsim v_n$ and therefore there exists a sequence $\{w_n\}$ such that $a_n \sim w_n \geq v_n$; the conclusion $a_n \gtrsim c_n$ is now immediate.

Exercise 2.11. Conclude from the preceding exercises that the "$\gtrsim$" relation is a partial order on the set of asymptotic equivalence classes of sequences of real numbers.

Exercise 2.12. Prove: $a_n \gtrsim 0$ if an only if $(\exists n_0)(\forall n \geq n_0)(a_n \geq 0)$, i. e., $a_n \geq 0$ for all sufficiently large n.

Hint: Exercises 2.9 and 1.2.

Next we turn to sequences of positive numbers; in this case, the following more intuitive characterization of the $\gtrsim$ relation be given.

Exercise 2.13. Let $\{a_n\}$ and $\{b_n\}$ be sequences of positive numbers. Prove: $a_n \gtrsim b_n$ if and only if $\liminf a_n/b_n \geq 1$.

An equivalent version of this characterization is the following:

Exercise 2.14. Let $a_n, b_n \geq 0$. Prove: $a_n \gtrsim b_n$ if and only if there exists a sequence $\{\epsilon_n\}$ of positive numbers such that $\epsilon_n \to 0$ and $(\forall n)\,(a_n \geq (1-\epsilon_n)b_n)$.

Yet another equaivalent version:

Exercise 2.15. Let $a_n, b_n \geq 0$. Prove: $a_n \gtrsim b_n$ if and only if
$(\forall \epsilon > 0)(\exists n_0)(\forall n > n_0)\,(a_n \geq (1-\epsilon)b_n)\,.$

Exercise 2.16. Show that the formula given in the preceding exercise does NOT define the relation "$a_n \gtrsim b_n$" if we omit the condition $a_n, b_n \geq 0$.

Exercise 2.17. Prove: If $a_n \gtrsim b_n \geq 0$ and $c_n \gtrsim d_n \geq 0$ then $a_n c_n \gtrsim b_n d_n$.

Exercise 2.18. Prove: If $a_n \gtrsim b_n \geq 0$ and $c_n \gtrsim d_n \geq 0$ then $a_n + c_n \gtrsim b_n + d_n$.

Exercise 2.19. Assume $a_n, b_n > 1$. Consider the following statements:

(A) $a_n \gtrsim b_n$;

(B) $\ln a_n \gtrsim \ln b_n$.

Prove:

(a) (A) does not imply (B).

(b) (A) does imply (B) under the stronger assumption that a_n is bounded away from 1. (Cf. Exercise 1.10.)

Exercise$^+$ 2.20. Let $a_n \geq 1$. Prove: $a_n^2 \ln a_n \gtrsim n$ if and only if $a_n \gtrsim \sqrt{2n/\ln n}$.

Exercise 2.21. We are given n distinct weights and want to sort them using a balance to compare pairs of weights. The weights are given in any order, so there are $n!$ possible inputs.

(a) Suppose an algorithm sorts every input using $\leq C(n)$ comparisons. Prove: $C(n) \gtrsim n \log_2 n$.

(b) Prove the same conclusion if the algorithm is only required to work correctly one percent of the time, i. e., it will correctly sort $n!/100$ inputs.