1 Limit

We begin with reviewing the concept of a limit.

**Definition 1.1** (Finite limit). Let \( \{a_n\} \) be a sequence\(^1\) and \( L \) a (real or complex) number. The statement \( \lim_{n \to \infty} a_n = L \), also denoted \( a_n \to L \), means that

\[
(\forall \epsilon > 0)(\exists n_0)(\forall n > n_0)(|a_n - L| < \epsilon).
\]

(1)

In other words, for all positive values \( \epsilon \), we have \( |a_n - L| < \epsilon \) for all sufficiently large values of \( n \).

Yet in other words, for all positive values of \( \epsilon \), the quantity \( |a_n - L| \) is eventually less than \( \epsilon \).

We use the word “eventually” to mean that something happens for all sufficiently large values of the parameter (in this case, \( n \)). The formal meaning of the statement “something happens for all sufficiently large values of \( n \)” is that there exists a threshold \( n_0 \) such that for all \( n \) beyond that threshold, the claimed “something” happens. Here is an example.

**Definition 1.2.** Let \( \{a_n\} \) be a sequence. We say that this sequence is **eventually zero** if \( a_n = 0 \) for all sufficiently large \( n \). In formula, this means that

\[
(\exists n_0)(\forall n > n_0)(a_n = 0).
\]

**Exercise 1.3.** (a) Define what it means for a sequence to be eventually nonzero.

(b) Find a sequence that is neither eventually zero nor eventually nonzero.

**Remark 1.4.** Note that the limit defined in Eq. (1) is not affected if we change a finite number of terms in the sequence. The definition is not affected even if a finite number of terms is undefined.

**Exercise 1.5.** Equation (1) defines *finite* limits (\( L \) is a number). Infinity is NOT a number. Give a definition, analogous to Eq. (1), that defines the statement that \( \lim a_n = \infty \).

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\(^1\) By a *sequence*, in this note we always mean an infinite sequence of real or complex numbers.
2 Standard definition and examples of asymptotic equality

Often, we are interested in comparing the rate of growth of two functions, as inputs increase in length. Asymptotic equality is one formalization of the idea of two functions having the “same rate of growth.”

Let \( \{a_n\} \) and \( \{b_n\} \) be two sequences.

**Definition 2.1** (Standard definition of asymptotic equality). We say that \( a_n \) is asymptotically equal to \( b_n \) if \( \lim_{n \to \infty} \frac{a_n}{b_n} = 1 \). We denote this circumstance by \( a_n \sim b_n \).

Warning: asymptotic equality does not mean eventual equality, as, for instance, example (a) in the next exercise shows.

**Exercise 2.2.**  
(a) Prove: \( n^2 + n \sim n^2 \)

(b) Prove: \( 5n^3 + 7n - 1000 \sim 5n^3 \)

(c) Prove: \( \frac{5n^3 + 7n - 1000}{8n^{10} - 6n^2 - 9n + 7} \sim \frac{5}{8n^7} \)

(d) Note that for \( n = 1 \), the denominator in the preceding example is zero. Explain why this is not a problem.

**Exercise 2.3.**  
1. If \( f(x) \) and \( g(x) \) are polynomials with respective leading terms \( ax^k \) and \( bx^\ell \) then \( f(n)/g(n) \sim (a/b)x^{k-\ell} \).

2. \( \sin(1/n) \sim 1/n \).

3. \( \ln(1 + 1/n) \sim 1/n \).

4. \( \sqrt{n^2 + 1} - n \sim 1/(2n) \).

**Exercise 2.4.**  
(a) If \( f \) is a function, differentiable at zero, \( f(0) = 0 \), and \( f'(0) \neq 0 \), then \( f(1/n) \sim f'(0)/n \).

(b) Show that items 2—4 of the preceding exercise follow from part (a) of this exercise.

**Exercise 2.5.** Find two sequences of positive real numbers, \( \{a_n\} \) and \( \{b_n\} \), such that \( a_n \sim b_n \) but \( a_n^m \neq b_n^m \).

Next we state some of the most important asymptotic formulas in mathematics.

**Theorem 2.6** (Stirling’s Formula).

\[ n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}. \]

**Exercise 2.7.** Prove: \( \left(\frac{2n}{n}\right)^n \sim \frac{4^n}{\sqrt{\pi n}} \).
Next we state one of the most beautiful theorems of all of mathematics.

**Definition 2.8** (Prime counting function). Let \( \pi(x) \) denote the number of primes less than or equal to \( x \).

So \( \pi(10) = 4, \pi(100) = 25, \pi(\pi) = 2, \pi(-10) = 0 \).

**Theorem 2.9** (The Prime Number Theorem).

\[ \pi(x) \sim \frac{x}{\ln x}, \]

where \( \ln \) denotes the natural logarithm function.

This fundamental result was proved in 1896 independently by Jacques Hadamard and Charles de la Vallée Poussin, using the theory of complex functions (Riemann’s zeta function).

**Exercise 2.10. Feasibility of generating random prime numbers.** Estimate, how many random \( \leq 100 \)-digit integers should we expect to pick before we encounter a prime number. (We generate our numbers by choosing the 100 digits independently at random (initial zeros are permitted), so each of the \( 10^{100} \) numbers has the same probability to be chosen.) Interpret this question as asking the reciprocal of the probability that a randomly chosen integer is prime. Assume, for the purposes of this estimate, that \( 10^{100} \) is sufficiently large for the Prime Number Theorem to give a good estimate. State how good the estimate needs to be.

## 3 Properties of asymptotic equality

The following simple exercise has great conceptual significance.

**Exercise 3.1.** If \( a_n \sim b_n \) then both of these sequences are eventually nonzero.

Let \( c \) be a (real or complex) number. We write \( c \) to denote the sequence \( (c, c, \ldots) \).

**Exercise 3.2.** If \( c \neq 0 \) is a number then the statement \( a_n \sim c \) is equivalent to \( a_n \to c \).

Note that this is false for \( c = 0 \): the sequence 0 is not asymptotically equal to any sequence, not even itself, according to Ex. 3.1.

**Exercise 3.3.** Let \( S \) denote the set of eventually nonzero sequences. Prove that \( \sim \) is an equivalence relation on \( S \), i.e., the relation “\( \sim \)” is

(a) reflexive: \( a_n \sim a_n \);
(b) symmetric: if \( a_n \sim b_n \) then \( b_n \sim a_n \); and
(c) transitive: if \( a_n \sim b_n \) and \( b_n \sim c_n \) then \( a_n \sim c_n \).
Implicit universal quantifiers: The statements above hold for all sequences \( \{a_n\}, \{b_n\}, \{c_n\} \in S \).

**Exercise 3.4.** Prove: if \( a_n \sim b_n \) and \( c_n \sim d_n \) then \( a_n c_n \sim b_n d_n \) and \( a_n / c_n \sim b_n / d_n \). (Recall that a finite number of undefined terms do not invalidate a limit relation.)

**Exercise 3.5.** Consider the following statement.

If \( a_n \sim b_n \) and \( c_n \sim d_n \) then \( a_n + c_n \sim b_n + d_n \). \( \text{(2)} \)

1. Prove that (2) is false, even if we assume that all the six sequences involved are eventually nonzero.
2. Prove: if \( a_n c_n \sim 0 \) for all sufficiently large \( n \) then (2) is true.

**Hint.** Prove: if \( a, b, c, d > 0 \) and \( a/b < c/d \) then \( a/b < (a+c)/(b+d) < c/d \).

We say that the “statement \( A \) implies statement \( B \)” if \( B \) follows from \( A \).

**Exercise 3.6.** Assume \( a_n, b_n > 1 \). Consider the following statements:

(A) \( a_n \sim b_n \);

(B) \( \ln a_n \sim \ln b_n \).

Prove:

(a) (A) does not imply (B).

(b) (A) does imply (B) under the stronger assumption that \( a_n \geq 1.01 \).

**Definition 3.7.** We say that the sequence \( a_n \) is bounded away from the number \( L \) if there exists \( c > 0 \) such that for all sufficiently large \( n \) we have \( |a_n - L| > c \).

The condition \( a_n \geq 1.01 \) in the preceding exercise can be replaced by the condition that \( a_n \) is bounded away from 1 (while we continue to assume \( a_n > 1 \)).

**Exercise 3.8** (Squeeze principle). Assume that for all sufficiently large \( n \) we have \( a_n \leq b_n \leq c_n \). Assume further that \( a_n \sim c_n \). Prove: \( a_n \sim b_n \sim c_n \).

**Hint.** Use the power series expansion of \( e^x \).

Note that (a) this result does not follow from Stirling’s formula; but (b) it follows from Stirling’s formula for all sufficiently large \( n \).

**Exercise 3.9.** Prove: for all \( n \geq 1 \) we have \( n! > \left( \frac{n}{e} \right)^n \).

**Hint.** Use the power series expansion of \( e^x \).

**Exercise 3.10.** Prove: \( \ln(n!) \sim n \ln n \). Give two proofs. (1) Use Stirling’s formula. (2) Do not use Stirling’s formula; use Ex. 3.9 instead.

**Exercise 3.11.** Let \( p_n \) be the \( n \)-th prime number. Consider the following statement: \( p_n \sim n \ln n \). Prove that this statement is equivalent to the Prime Number Theorem.

**Exercise** \( +3.12. \) Let \( P(x) \) denote the product of all prime numbers \( \leq x \). Consider the following statement: \( \ln P(x) \sim x \). Prove that this statement is equivalent to the Prime Number Theorem.
4 Extended definition of asymptotic equality

We wish the relation of asymptotic equality to be an equivalence relation among all sequences. This is not true under the standard definition; the relation is not even reflexive: the all-zero sequence 0 is not asymptotically equal to itself.

Exercise 4.1. \(a_n \sim a_n\) if and only if the sequence \(\{a_n\}\) is eventually nonzero.

To remedy this, we shall replace every occurrence of the fraction \(0/0\) by 1.

Remark 4.2. This does not mean that we think \(0/0 = 1\). No, the fraction \(0/0\) continues to be undefined. This is simply a technical trick, we replace all occurrences of the fraction \(0/0\) among the fractions \(a_n/b_n\) by 1. The benefits of this trick will be apparent below. I note that this is not a standard trick, you will not find it in any textbook.

So the exact definition goes as follows.

Definition 4.3 (Extended definition of asymptotic equality). Let \(\{a_n\}\) and \(\{b_n\}\) be two sequences. Let \(q_n = a_n/b_n\) except when \(a_n = b_n = 0\); in that case, let \(q_n = 1\). We say that the sequence \(\{a_n\}\) is asymptotically equal to the sequence \(\{b_n\}\) under the extended definition if \(\lim_{n \to \infty} q_n = 1\). We denote this circumstance by \(a_n \sim b_n\).

Note that if \(a_n \neq 0\) but \(b_n = 0\) then \(q_n\) is undefined; and if infinitely many terms of the sequence \(\{q_n\}\) are undefined then the sequence has no limit.

Exercise 4.4. Our extended definition indeed extends the standard definition:

(a) If \(a_n \sim b_n\) then \(a_n \sim b_n\).

(b) If at least one of the sequences \(\{a_n\}\) and \(\{b_n\}\) is eventually nonzero, then \(a_n \sim b_n\) if and only if \(a_n \sim b_n\).

The converse of item (a) is not true: \(0 \sim 0\) but \(0 \not\sim 0\).

Compare item (b) with Ex. 3.1.

5 Properties of extended asymptotic equality

From now on we use our extended definition of asymptotic equality, Def. 4.3. Remember, however, that if we restrict our attention to eventually nonzero sequences then there is no difference between the two concepts.

Exercise 5.1. Prove: \(a_n \sim 0\) if and only if \(a_n\) is eventually zero.

Exercise 5.2. Let \(S^*\) denote the set of all sequences of real or complex numbers. Prove that \(\sim\) is an equivalence relation on \(S^*\), i.e., the relation \(\sim\) is

(a) reflexive: \(a_n \sim a_n\);
(b) symmetric: if \( a_n \sim b_n \) then \( b_n \sim a_n \); and

(c) transitive: if \( a_n \sim b_n \) and \( b_n \sim c_n \) then \( a_n \sim c_n \).

Implicit universal quantifiers: The statements above hold for all sequences \( \{a_n\}, \{b_n\}, \{c_n\} \).

Exercise 5.3. Prove: if \( a_n \sim b_n \) and \( c_n \sim d_n \) then \( a_n c_n \sim b_n d_n \). If, moreover, \( c_n d_n \neq 0 \) for all sufficiently large \( n \) then \( a_n / c_n \sim b_n / d_n \).

6 Asymptotic inequality

We wish to define a notion of asymptotic inequality between sequences, to be called the “greater than or asymptotically equal” relation, denoted \( a_n \gtrsim b_n \). It is natural to expect our definition to satisfy the following conditions:

(a) \( a_n \sim b_n \) if and only if both \( a_n \gtrsim b_n \) and \( b_n \gtrsim a_n \) hold;

(b) if \( (\forall n)(a_n \geq b_n) \) then \( a_n \gtrsim b_n \); and

(c) if \( a_n \gtrsim b_n \) and \( a_n \gtrsim c_n \) then \( a_n \gtrsim \max\{b_n, c_n\} \).

It turns out that these three conditions together already determine the concept. The simplest definition we could find that works for all pairs of sequences is described next. A perhaps more intuitive definition that works in the important case of sequences of positive numbers will be given in Exercise *** below. The definition will be followed by a series of exercises each of which can be solved in a few lines given the exercises preceding it; these exercises reveal the basic properties of the \( \gtrsim \) relation.

Definition 6.1. Let \( \{a_n\} \) and \( \{b_n\} \) be sequences of real numbers. We say that \( a_n \) is greater than or asymptotically equal to \( b_n \), denoted as \( a_n \gtrsim b_n \) if \( a_n \sim \max\{a_n, b_n\} \).

A dual definition, using \( \min \), can also be given; the two definitions are equivalent (Exercise 6.6). Before proving the equivalence, we verify some simple consequences of the definition given.

Exercise 6.2. Prove: if \( a_n \sim b_n \) then \( a_n \gtrsim b_n \).

Exercise 6.3. Prove: if \( a_n \geq b_n \) holds for all sufficiently large \( n \) then \( a_n \gtrsim b_n \).

Exercise 6.4. (a) Prove: if \( a_n \gtrsim b_n \) and \( b_n \gtrsim a_n \) then \( a_n \sim b_n \).

(b) [Squeeze principle] Prove: if \( a_n \lesssim b_n \lesssim c_n \) and \( a_n \sim c_n \) then \( a_n \sim b_n \sim c_n \).

(c) Prove: if \( a_n \lesssim b_n \lesssim c_n \) and \( a_n \sim c_n \) then \( a_n \sim b_n \sim c_n \).

These facts are immediate from the definition. Now we give a somewhat technical yet intuitive equivalent definition.
Exercise 6.5. Let \{a_n\} and \{b_n\} be sequences of real numbers. Let \(B = \{n : a_n < b_n\}\). We claim that \(a_n \gtrsim b_n\) if and only if either \(B\) is finite or the subsequences \(\{a_n : n \in B\}\) and \(\{b_n : n \in B\}\) are asymptotically equal (in the extended sense).

This exercise seems to justify the term “greater than or asymptotically equal” for some of the subscripts, \(a_n \geq b_n\); and for the remaining subscripts, \(a_n \sim b_n\).

The dual characterization of the \(\gtrsim\) relation is now immediate.

Exercise 6.6. Prove: \(a_n \gtrsim b_n\) if and only if \(b_n \sim \min\{a_n, b_n\}\).

The following is an immediate corollary:

Exercise 6.7. \(a_n \gtrsim b_n\) if and only if \(-b_n \gtrsim -a_n\).

We begin our preparations for proving that the \(\gtrsim\) relation is transitive.

Exercise 6.8. Prove: \(a_n \gtrsim b_n\) if and only if there exists a sequence \(d_n\) such that \(a_n \sim d_n \geq b_n\).

Note that the \(\Rightarrow\) direction is immediate from the definition; the \(\Leftarrow\) direction follows using Exercise 6.5.

Exercise 6.9. Prove: \(a_n \gtrsim b_n\) if and only if there exists a sequence \(c_n\) such that \(a_n \geq c_n \sim b_n\).

This exercise is an immediate consequence of the preceding exercise by switching signs and using Exercise 6.7.

Now we are ready to prove transitivity.

Exercise 6.10. Prove: if \(a_n \gtrsim b_n\) and \(b_n \gtrsim c_n\) then \(a_n \gtrsim c_n\).

The proof only requires some manipulation of symbols using the preceding two exercises. Indeed, notice that there exist sequences \(\{u_n\}\) and \(\{v_n\}\) such that \(a_n \sim u_n \sim v_n \geq c_n\). It follows that \(a_n \gtrsim v_n\) and therefore there exists a sequence \(\{w_n\}\) such that \(a_n \sim w_n \geq v_n\); the conclusion \(a_n \gtrsim c_n\) is now immediate.

Exercise 6.11. Conclude from the preceding exercises that the “\(\gtrsim\)” relation is a partial order on the set of extended asymptotic equivalence classes of sequences of real numbers.

Exercise 6.12. Prove: \(a_n \gtrsim 0\) if and only if \(a_n\) is eventually nonnegative, i.e., \(a_n \geq 0\) for all sufficiently large \(n\).

Hint: Exercises 6.9 and 5.1.

Next we turn to sequences of positive numbers; in this case, the following more intuitive characterization of the \(\gtrsim\) relation is given.

Exercise 6.13. Let \(\{a_n\}\) and \(\{b_n\}\) be sequences of positive numbers. Prove: \(a_n \gtrsim b_n\) if and only if \(\liminf a_n/b_n \geq 1\).
An equivalent version of this characterization is the following:

**Exercise 6.14.** Let \( a_n, b_n \geq 0 \). Prove: \( a_n \gtrsim b_n \) if and only if there exists a sequence \( \{ \epsilon_n \} \) of positive numbers such that \( \epsilon_n \to 0 \) and \( \forall n \) \( (a_n \geq (1 - \epsilon_n) b_n) \).

Yet another equivalent version:

**Exercise 6.15.** Let \( a_n, b_n \geq 0 \). Prove: \( a_n \gtrsim b_n \) if and only if there exists a sequence \( \{ \epsilon_n \} \) of positive numbers such that \( \epsilon_n \to 0 \) and \( \forall n \) \( (a_n \geq (1 - \epsilon_n) b_n) \).

**Exercise 6.16.** Show that the formula given in the preceding exercise does NOT define the relation “\( a_n \gtrsim b_n \)” if we omit the condition \( a_n, b_n \geq 0 \).

**Exercise 6.17.** Prove: If \( a_n \gtrsim b_n \) and \( c_n \gtrsim d_n \) and \( \{ b_n \} \) and \( \{ d_n \} \) are eventually nonnegative then \( a_n c_n \gtrsim b_n d_n \).

**Exercise 6.18.** (a) Prove: If \( a_n - b_n \gtrsim 0 \) then \( a_n \gtrsim b_n \).

(b) The converse is false: \( a_n \gtrsim b_n \) does not imply \( a_n - b_n \gtrsim 0 \).

**Exercise 6.19.** (a) Prove: If \( a_n \gtrsim b_n \) and \( c_n \gtrsim d_n \) and \( \{ b_n \} \) and \( \{ d_n \} \) are eventually nonnegative then \( a_n c_n \gtrsim b_n d_n \).

(b) Show that the conclusion would not follow without the nonnegativity assumption.

**Exercise 6.20.** Assume \( a_n, b_n > 1 \). Consider the following statements:

(A) \( a_n \gtrsim b_n \);

(B) \( \ln a_n \gtrsim \ln b_n \).

Prove:

(a) (A) does not imply (B).

(b) (A) does imply (B) under the stronger assumption that \( a_n \) is bounded away from 1. (Cf. Exercise 3.6.)

**Exercise+ 6.21.** Let \( a_n \geq 1 \). Prove: \( a_n^2 \ln a_n \gtrsim n \) if and only if \( a_n \gtrsim \sqrt{2n/\ln n} \).

**Exercise 6.22.** We are given \( n \) distinct weights and want to sort them using a balance to compare pairs of weights. The weights are given in any order, so there are \( n! \) possible inputs.

(a) Suppose an algorithm sorts every input using \( \leq C(n) \) comparisons. Prove: \( C(n) \gtrsim n \log_2 n \).

(b) Prove the same conclusion if the algorithm is only required to work correctly one percent of the time, i.e., it will correctly sort \( n!/100 \) inputs.

**Exercise 6.23.** (a) Find a sequence \( \{ a_n \} \) of positive numbers such that \( a_{n+1} \gtrsim a_n \) and \( a_n \to 0 \).

(b) Let \( \{ a_n \} \) be a sequence of positive numbers such that \( a_{n+1} \gtrsim a_n \). Show that \( a_n \) cannot decrease exponentially, i.e., for all \( 0 < c < 1 \), we have \( a_n > c^n \) for all sufficiently large \( n \).