Honors Graph Theory – CMSC-27530 = Math 28530

Independence number of Erdős–Rényi random graphs

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Abstract

We describe an upper bound on the independence number Erdős–Rényi random graphs and two results of Erdős that follow from this bound: (a) an exponential lower bound on the diagonal Ramsey numbers, and (b) the existence of small triangle-free graphs of high chromatic number. The order of the latter will be $n = O((k \ln k)^3)$ where k is the desired chromatic number.

1 Upper bound on the independence number whp

Let us pick the graph $\mathcal{G} = ([n], E)$ from the $\mathbf{G}(n, p)$ model. We denote this circumstance by $\mathcal{G} \sim \mathbf{G}(n, p)$.

Let $A \subseteq [n]$, $|A| = k$.

The probability that A is independent in $\mathcal{G}$ is $(1-p)^{\binom{k}{2}}$.

Therefore, by the union bound and using the trivial inequality $\binom{n}{k} \le n^k/k!$, the probability that $\alpha(\mathcal{G}) \ge k$ is less than

$$\binom{n}{k}(1-p)^{\binom{k}{2}} < \frac{1}{k!}\left(n(1-p)^{(k-1)/2}\right)^k . \tag{1}$$

This proves part (a) of the following statement.

Proposition 1.1. *(a)* *If* $n(1-p)^{(k-1)/2} \le 1$ *then* $P(\alpha(\mathcal{G}) \ge k) < \frac{1}{k!}$.
(b) *If* $\ln n \le p(k-1)/2$ *then* $P(\alpha(\mathcal{G}) \ge k) < \frac{1}{k!}$.

Part (b) follows from Part (a) in the light of the inequality $1 + x \le \mathrm{e}^x$, applied to $x := -p$.
Part (b) is useful when p is small.

2 Exponential lower bound on the diagonal Ramsey numbers

Recall that the *Erdős–Rado arrow symbol* $n \to (k, \ell)$ ("n arrows (k, ℓ)") denotes the statement that every graph G of order n satisfies $\alpha(G) \ge k$ or $\omega(G) \ge \ell$, where $\omega(G)$ denotes the clique number of G. The *Ramsey number* $R(k, \ell)$ is the smallest value n such that $n \to (k, \ell)$. The *diagonal Ramsey numbers* refer to the case $k = \ell$.

Theorem 2.1 (Erdős, 1947)**.**

$$R(k+1, k+1) > 2^{k/2} .$$

Compare this with the Erdős–Szekeres upper bound (1935):

$$R(k+1,k+1) < \binom{2k}{k} < 2^{2k} = 4^k\,. \tag{2}$$

In spite of significant efforts, the base 4 was not improved in this upper bound for 88 years. On March 16, 2023, a paper was posted on arXiv, "An exponential improvement for diagonal Ramsey" by Marcelo Campos, Simon Griffiths, Robert Morris, and Julian Sahasrabudhe, arXiv 2303.09521, claiming an upper bound of $(4-c)^k$ for some positive constant c. The result, when verified, will count as a major breakthrough.
No similar improvement of the base $\sqrt{2}$ in Erdős's lower bound $2^{k/2} = \sqrt{2}^k$ (Theorem 2.1) is known. Also wide open is Erdős's conjecture that $\lim_{k\to\infty} R(k,k)^{1/k}$ exists; all we know is that the lower limit is at least $\sqrt{2}$ (Theorem 2.1) and the upper limit is at most $4-c$ (the recent paper).

Proof of Theorem 2.1. Let us set $p = 1/2$. Then, from Prop. 1.1, we get that if $n \le 2^{(k-1)/2}$ then

$$P(\omega(\mathcal{G}) \ge k) = P(\alpha(\mathcal{G}) \ge k) \le \frac{1}{k!} \tag{3}$$

and therefore the probability of the OR of these two events is less than $2/k!$. It follows that there exists a graph of order $2^{k/2}$ that has no clique and no independent set of size $k+1$, i. e.,

$$2^{k/2} \not\to (k+1,k+1)\,. \tag{4}$$

By definition this is equivalent to Theorem 2.1. □

Remark 2.2. We call a subset $A \subseteq V$ *homogeneous* in the graph $G = (V,E)$ if either A is an independent set in G or A induces a clique in G.

Not only did we show that there exists a graph of order $n = 2^{k/2}$ without homogeneous subsets of size $k+1$; we have shown that *almost all* graphs have this property. Yet an explicit construction of a graph with $(1+c)^k$ without a homogeneous subset of size $k+1$ remain open, although much progress has been made. It seems the current champion has order $n = \exp(k^{1/(\log\log k)^c})$ by Gil Cohen,
"Towards optimal two-source extractors and Ramsey graphs," *Electr. Colloq. Computat. Complexity* 114 (2016).

3 Small triangle-free graphs with large chromatic number

In this section we prove the existence of small triangle-free graphs of large chromatic number, one of the early triumphs of the Probabilistic Method.

Theorem 3.1 (Erdős 1957)**.** *For every k there exists a triangle-free graph of order $O((k\ln k)^3)$ and chromatic number k.*

Compare this with Mycielski's triangle-free graphs of large chromatic number; those graphs have order $3\cdot 2^{k-2}-1$ for chromatic number k. This order grows *exponentially* while Erdős's grows *polynomially* as a function of the desired chromatic number k.

For the proof we use the $\mathbf{G}(n,p)$ model, where p will be a function of n. Let $\mathcal{G} \sim \mathbf{G}(n,p)$.

Let X_3 denote the number of triangles in $\mathcal{G}$. Then

$$E(X_3) = p^3 \binom{n}{3} < (np)^3/6. \tag{5}$$

Lemma 3.2. *If p is chosen such that*

(a) $p \le cn^{-2/3}$ *where* $c = (3/2)^{1/3}$ *and*

(b) $\ln n \le p(\ell - 1)/2$

then there exists a triangle-free graph with at most n vertices and chromatic number $\ge n/(2\ell)$.

Proof. Picking $\mathcal{G}$ from $\mathbf{G}(n,p)$, we have $E(X_3) \le n/4$ and $P(\alpha(\mathcal{G}) \ge \ell) < 1/\ell!$. By Markov's inequality, $P(X_3 \ge n/2) \le 1/2$ and therefore the probability that $X_3 < n/2$ and $\alpha(\mathcal{G}) < \ell$ is at least $1/2 - 1/\ell! > 0$. Therefore both of these occur for some graph G. Let us now delete a vertex from each triangle; we have deleted fewer than $n/2$ vertices. The remaining graph H therefore has $n' > n/2$ vertices, is triangle-free, and satisfies $\alpha(H) < \ell$. Therefore $\chi(H) \ge n'/\alpha(H) > n/(2\ell)$. □

Now we are ready to prove the existence of small triangle-free graphs of large chromatic number.

Proof of Theorem 3.1. Let us set $p = cn^{-2/3}$ where $c = (3/2)^{1/3} = 1.1447\ldots > 8/7$. This p satisfies condition (a) in Lemma 3.2.
We want to satisfy condition (b). For $\ell \ge 8$ we have $p(\ell - 1) \ge n^{-3/2}\ell$. Therefore it suffices to guarantee that $\ln n \le n^{-2/3}\ell/2$, in other words, $2n^{2/3}\ln n \le \ell$. So let us choose $\ell = 2n^{2/3}\ln n$. Now we get a triangle-free graph with $\le n$ vertices and chromatic number $k \ge n/(2\ell) = n^{1/3}/(4\ln n)$. From this we get $n \lesssim (12k \ln k)^3$. □