Definitions 1.1.  

- An **abelian group** is a set $G$ with the following properties:
  
  (i) $(\forall a, b \in G)(\exists! a + b \in G)$
  
  (ii) the addition in (i) is associative
  
  (iii) $(\exists 0)(\forall a \in G)(a + 0 = a)$
  
  (iv) $(\forall a)(\exists b)(a + b = 0)$
  
  (v) $a + b = b + a$

- A **vector space** is an abelian group $(V, +)$ with a multiplication by scalars:
  
  $- (\forall \alpha \in \mathbb{R})(\forall a \in V)(\exists! \alpha a \in V)$
  
  $- (\alpha \beta) a = \alpha (\beta a)$
  
  $- (\alpha + \beta) a = \alpha a + \beta a$
  
  $- \alpha (a + b) = \alpha a + \alpha b$

**Theorem 1.2.** $\alpha a = 0 \iff \alpha = 0$ or $a = 0$.

**Corollary 1.3.** $(-1)a = -a$.

**Examples 1.4.** Some examples of vector spaces.

1. $\mathbb{R}^n$
2. geometric vectors in 2 or 3 dimensions
3. $k \times \ell$ matrices
4. $C[0, 1]$, the continuous functions from $[0, 1]$ to $\mathbb{R}$
5. the space of infinite sequences
6. $\mathbb{R}^\Omega$, the functions from $\Omega$ to $\mathbb{R}$
• $\Omega = \{1, \ldots, n\}$ is example 1 above
• $\Omega = \{1, \ldots, k\} \times \{1, \ldots, \ell\}$ is example 3
• $\Omega = [0, 1]$ contains example 4
• $\Omega = \mathbb{N}$ is example 5

**Definitions 1.5.**

- A **linear combination** of $a_1, \ldots, a_k \in V$ is a sum $\sum_{i=1}^{k} \alpha_i a_i$ for some $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$.
- The **span** of $a_1, \ldots, a_k$, written $\text{Span}(a_1, \ldots, a_k)$, is the set of all linear combinations of $a_1, \ldots, a_k$. More generally, for a possibly infinite subset $S$ of $V$, $\text{Span}(S)$ is the set of all linear combinations of all finite subsets. By convention, $\text{Span}(\emptyset) = \{0\}$.

- A **subspace** of $V$ is a subset $W$ which is a vector space under the same operations. This is written $W \leq V$. Equivalently, a subset $W$ is a subspace if it is nonempty and closed under addition and multiplication by scalars:
  1. $W \neq \emptyset$
  2. $(\forall a, b \in W)(a + b \in W)$
  3. $(\forall a \in W)(\forall \lambda \in \mathbb{R})(\lambda a \in W)$.

  Equivalently, a subspace is a nonempty subset closed under linear combinations.

**Corollary 1.6.** $W \leq V$ iff $\text{Span}(W) = W$.

**Exercise 1.7.** If $S \subseteq W$, then $\text{Span}(S) \subseteq W$.

- The vectors $a_1, \ldots, a_k$ are **linearly independent** if $(\forall \alpha_1, \ldots, \alpha_k \in \mathbb{R})(\sum_{i=1}^{k} \alpha_i a_i = 0 \Rightarrow \alpha_1 = \cdots = \alpha_k = 0)$. An infinite set of vectors is linearly independent if all finite subsets are linearly independent. If a set of vectors is not linearly independent, it is **linearly dependent**.

**Examples 1.8.** Some more examples of vector spaces.

- $\mathbb{R}[x] = \{\text{polynomials with real coefficients}\}$.
- $\mathbb{R}(x) = \{\text{rational functions with real coefficients}\}$. That is, the set of $\frac{p}{q}$ with $p, q \in \mathbb{R}[x]$, $q \neq 0$, and $\frac{p_1}{q_1} = \frac{p_2}{q_2}$ iff $p_1 q_2 = p_2 q_1$.

**Claim 1.9.** $\{\frac{1}{x-\alpha} : \alpha \in \mathbb{R}\}$ is linearly independent in $\mathbb{R}(x)$.

**Definitions 1.10.**

- Let $S \subseteq V$. The **rank** of $S$, $\text{rk}(S)$, is the maximum number of linearly independent vectors in $S$.
- Let $W \leq V$. The **dimension** of $W$, $\dim(W)$, is the rank of $W$.
- Let $B \subseteq W \leq V$. $B$ is a **basis** of $W$ if (i) $\text{Span}(B) = W$ and (ii) $B$ is linearly independent.
A vector \( a \) depends on a set \( S \subseteq V \) if \( a \in \text{Span}(S) \).

**Corollary 1.11.** \( \{a_1, \ldots, a_k\} \) is a basis of \( W \) if \((\forall w \in W) (\exists! \alpha_1, \ldots, \alpha_k \in \mathbb{R}) (\sum \alpha_i a_i = w)\).

If \( \{a_1, \ldots, a_k\} \) is a basis for \( W \) and \( w \in W \), then the coordinates of \( w \) with respect to the basis \( \{a_1, \ldots, a_k\} \) is the column vector of the unique \( \alpha_i \) given by the corollary; we write

\[
[w]_{\{a_1, \ldots, a_k\}} = \begin{bmatrix}
\alpha_1 \\
\vdots \\
\alpha_k
\end{bmatrix}.
\]

The coordinates of \( w \) depend on the choice of basis for \( W \). Later we shall see how coordinate vectors change under change of basis.

**Observation 1.12.** A set of vectors \( S \) is linearly dependent iff there is a member which depends on the rest.

**Note 1.13.** A set containing \( 0 \) is never linearly independent. Also, a sequence of vectors with repetitions is never linearly independent.

**Exercise 1.14.** Prove that these functions are linearly independent in \( \mathbb{R}^R \): \( 1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx, \ldots \).

**Theorem 1.15 (Magic #1).** If \( u_1, \ldots, u_k \in \text{Span}(v_1, \ldots, v_\ell) \) and \( u_1, \ldots, u_k \) are linearly independent, then \( k \leq \ell \).

(Prove later.)

**Corollary 1.16.** If \( B_1 \) and \( B_2 \) are bases of \( W \), then \( |B_1| = |B_2| \).

**Exercise 1.17.** Every vector space has a basis. In fact, every set of generators contains a basis, and every linearly independent set can be extended to a basis.

**Exercise 1.18.** Let \( L \subseteq G \subseteq W \), and suppose \( \text{Span}(G) = W \). If \( L \) is linearly independent, then there is a basis \( B \) such that \( L \subseteq B \subseteq G \).

**Lemma 1.19.** If \( a_1, \ldots, a_k \) are linearly independent but \( a_1, \ldots, a_{k+1} \) are linearly dependent then \( a_{k+1} \in \text{Span}(a_1, \ldots, a_k) \).

The polynomials \( 1, x, x^2, \ldots \) are a basis for \( \mathbb{R}[x] \), showing that \( \dim \mathbb{R}[x] \) is countable. However, \( \dim \mathbb{R}(x) \) is uncountable since \( \{ \frac{1}{x-\alpha} : \alpha \in \mathbb{R} \} \) is linearly independent and uncountable.

A sequence \( (\alpha_0, \alpha_1, \alpha_2, \ldots) \) is a Fibonacci-type sequence if for \( n \geq 2 \), \( \alpha_n = \alpha_{n-1} + \alpha_{n-2} \). Let \( F \) be the set of all Fibonacci-type sequences. The Fibonacci-type sequence with \( \alpha_0 = 0 \) and \( \alpha_1 = 1 \) is called the Fibonacci sequence, \( f = (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots) \).

**Claim 1.20.** \( F \) is a 2-dimensional subspace of \( \mathbb{R}^N \).
A geometric sequence is one of the form \((1, q, q^2, \ldots)\). Can a geometric sequence be a Fibonacci-type sequence?

**Exercise 1.21.** The sequence \((1, q, q^2, \ldots)\) is Fibonacci-type iff \(1 + q = q^2\).

The equation \(q^2 = q + 1\) has two solutions: \(\frac{1 + \sqrt{5}}{2}\). The two geometric sequences \((1, \frac{1 + \sqrt{5}}{2}, \ldots)\) and \((1, \frac{1 - \sqrt{5}}{2}, \ldots)\) are linearly independent and, thus, a basis for \(F\).

**Corollary 1.22.** The Fibonacci sequence is a linear combination of these two geometric sequences.

**Exercise 1.23.** The \(n\)th term in the Fibonacci sequence is

\[
f_n = \frac{1}{\sqrt{5}} \left( \left(\frac{1 + \sqrt{5}}{2}\right)^n - \left(\frac{1 - \sqrt{5}}{2}\right)^n \right).
\]

**Exercise 1.24.**

\[
f_n = \left\lfloor \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n \right\rfloor
\]

where \(\lfloor x \rfloor\) means round \(x\) to the nearest integer.

**Definitions 1.25.** A \(k \times \ell\) matrix can be considered a set of \(\ell\) columns, \([a_1, \ldots, a_\ell]\), or a set of \(k\) rows, \([b_1, \ldots, b_k]\). The row space of the matrix is the span of these rows: \(\text{Span}(b_1, \ldots, b_k) \leq \mathbb{R}^k\). Similarly, the column space of the matrix is the span of the columns: \(\text{Span}(a_1, \ldots, a_\ell) \leq \mathbb{R}^k\). The row-rank of the matrix is the dimension of the row space, and the column-rank of the matrix is the dimension of the column space.

**Theorem 1.26 (Magic #2).** For any \(k \times \ell\) matrix, the column rank and the row rank are equal.

**Exercise 1.27.** Prove. (Do not use determinants).

**Fisher’s Inequality**

Let \(t \geq 1\). Let \(A_1, \ldots, A_m\) be subsets of \(\{1, \ldots, n\}\) such that

\[
(\forall i \neq j)(|A_i \cap A_j| = t).
\]

**Examples 1.28.**

- Let \(A_i = \{i, n\}\) for \(i = 1, \ldots, n - 1\) and \(A_n = \{1, \ldots, n - 1\}\).
- For \(n = 7\), the Fano Plane is a remarkable set of 7 subsets of size 3 of \(\{1, \ldots, 7\}\) with pairwise intersection size \(t = 1\).

**Theorem 1.29 (Fisher’s Inequality).** Condition \((\ref{1})\) implies \(m \leq n\).
Definition 1.30. For $A \subseteq \{1, \ldots, n\}$, define the incidence vector $v_A$ as $v_A = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ where

$$
\alpha_i = \begin{cases} 
1 & \text{if } i \in A \\
0 & \text{if } i \notin A 
\end{cases}
$$

Fisher's Inequality follows from Magic #1 and the following exercise.

Exercise 1.31. Let $a_1, \ldots, a_m$ be incidence vectors for sets $A_1, \ldots, A_m$ such that $(\forall i \neq j)(|A_i \cap A_j| = t)$. Then $a_1, \ldots, a_m$ are linearly independent in $\mathbb{R}^n$. 