6 Real Euclidean Spaces

Definition 6.1. If $V$ is a real vector space, an *inner product* on $V$ is:

$(\forall a, b \in V)(a \cdot b \in \mathbb{R})$

$(\forall \lambda \in \mathbb{R})(\forall a, b \in \mathbb{R})(\lambda a \cdot b = \lambda(a \cdot b))$

$(\forall a, b \in V)(a \cdot b = b \cdot a)$

$(\forall a, b, c \in V)(a \cdot (b + c) = a \cdot b + a \cdot c)$

$(\forall a \in V)\text{ s.t. } a \neq 0)(a \cdot a > 0)$

Note: We will frequently omit the $\cdot$ from $a \cdot b$ for convenience.

Definition 6.2. A *real Euclidean space* is a real vector space with an inner product.

Exercise 6.3. Show that if $V$ is a real Euclidean space and $a \in V$, then $a \cdot 0 = 0$.

Example 6.4. In $\mathbb{R}^2$ and $\mathbb{R}^3$, $a \cdot b = |a||b| \cos(\theta)$, where $\theta$ is the angle between the vectors $a$ and $b$, is an inner product. Note that distributivity (the fourth property listed in 6.1) in this example is highly nontrivial.

Example 6.5. The standard dot product is an inner product on $\mathbb{R}^n$. 
Example 6.6. A density function, \( \mu : \mathbb{R} \rightarrow \mathbb{R} \), is function such that

\[
(\forall x \in \mathbb{R})(\mu(x) \geq 0)
\]

\[
\int_{-\infty}^{+\infty} \mu(x)\,dx = 1
\]

\[
(\forall k \in \mathbb{N}) \left( \int_{-\infty}^{+\infty} x^{2k} \mu(x)\,dx < \infty \right).
\]

Given a density function, \( \mu \), we can define an inner product on \( \mathbb{R}[x] \) by

\[
f \cdot g = \int_{-\infty}^{+\infty} f(x)g(x)\mu(x)\,dx.
\]

- Hermite’s density function is \( \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \).
- Chebychev’s first density function is \( \frac{1}{\pi} \sqrt{1-x^2} \).
- Chebychev’s second density function is \( \frac{1}{\sqrt{\pi}} \sqrt{1-x^2} \).

In what follows, we let \( V \) be a real Euclidean space.

Definition 6.7. For \( a \in V \), the norm of \( a \) is

\[
||a|| = \sqrt{aa}.
\]

Using the norm, we can define a distance on \( V \).

Definition 6.8. For two vectors \( a, b \in V \), the distance between \( a \) and \( b \) is

\[
\text{dist}(a, b) = ||(b-a)||.
\]

Definition 6.9. Two vectors, \( a, b \in V \), are called orthogonal, written \( a \perp b \), if

\[
a \cdot b = 0.
\]

Definition 6.10. A set of vectors, \( \{a_i\} \subset V \), is called an orthogonal system of vectors if the \( a_i \) are pairwise orthogonal; i.e.,

\[
(\forall i, j)(a_i \perp a_j).
\]
**Exercise 6.11.** Prove that an orthogonal system of vectors is linearly independent.

**Definition 6.12.** An *orthogonal family of polynomials* with respect to $\mu$ is an orthogonal system in $\mathbb{R}[x]$ with inner product induced by the density function $\mu$.

**Exercise 6.13.** In $\mathbb{R}[x]$, let $f_0, f_1, f_2, \ldots$ be an orthogonal family of polynomials with respect to a density function $\mu$ such that $\deg(f_i) = i$. Show that the $f_i$ form a basis for $\mathbb{R}[x]$. Show furthermore that, for any density function, there is a unique such family.

**Exercise 6.14.** * Show that, in the exercise above, all the roots of the $f_i$ are real.

**Example 6.15.** The Hermite polynomials, $H_n$, are orthogonal with respect to the Hermite density function. The Chebyshev polynomials $T_n$ are orthogonal with respect to the first Chebyshev density and $U_n$ are orthogonal with respect to the second Chebyshev density.

The $T_n$ and $U_n$ are defined as functions of $\cos\theta$:

\[
T_0 = U_0 = 1
\]

\[
T_n(\cos\theta) = \cos n\theta
\]

\[
U_n(\cos\theta) = \frac{\sin((n + 1)\theta)}{\sin\theta}.
\]

Using the identity $\cos(2\theta) = 2\cos^2\theta - 1$ we have

\[
T_2(\cos\theta) = \cos(2\theta) = 2\cos^2\theta - 1.
\]

So as a function of $x$

\[
T_2(x) = 2x^2 - 1.
\]

Similarly, the identity $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$ yields

\[
T_3(x) = 4x^3 - 3x.
\]

Likewise, the identities

\[
\sin(2\theta) = 2\sin\theta\cos\theta
\]

\[
\sin(3\theta) = 3\cos^2\theta\sin\theta - \sin^3\theta = \sin\theta(4\cos^2\theta - 1)
\]
\[
\sin(4\theta) = 4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta = \sin \theta (8 \cos^3 \theta - 4 \cos \theta)
\]
yield

\[
U_1(x) = 2x
\]

\[
U_2(x) = 4x^2 - 1
\]

\[
U_3(x) = 8x^3 - 4x
\]

**Exercise 6.16.** Verify that these families are orthogonal with respect to the respective density functions.

### 6.1 Orthogonal Transformations

We now return to an arbitrary real Euclidean space, \(V\).

**Exercise 6.17.** Show that any finite dimensional real Euclidean space has an orthonormal basis

If \(e = (e_1, \ldots, e_n)\) is a basis for \(V\) and \(x \in V\), recall then the matrix of \(x\) with respect to \(e\) is written \([x]_e\).

If \(e\) is an orthonormal basis for \(V\), then the inner product on \(V\) is related to the standard dot product in \(\mathbb{R}^n\) via the basis \(e\).

**Theorem 6.18.** If \(e\) is an orthonormal basis for \(V\) and \(x, y \in \mathbb{R}\), then

\[
x \cdot y = [x]_e^T [y]_e
\]

where the left hand inner product is that of \(V\) and the right hand inner product is the dot product in \(\mathbb{R}^n\).

**Definition 6.19.** An orthogonal transformation is a distance-preserving map \(\varphi : V \to V\). That is,

\[
(\forall a, b \in V)(\text{dist}(\varphi(a), \varphi(b)) = \text{dist}(a, b)).
\]

**Exercise 6.20.** Check that any map \(\varphi : V \to V\) is an orthogonal transformation iff \(\varphi\) is norm-preserving; i.e.,

\[
(\forall a \in V)(\|\varphi(a)\| = \|a\|).
\]
Claim 6.21. Any orthogonal transformation $\varphi : V \to V$ preserves inner products.

Proof: For any $x, y \in V$ we have

$$\varphi(x + y) \cdot \varphi(x + y) = \|\varphi(x + y)\|^2 = \|x + y\|^2 = (x + y) \cdot (x + y)$$

since $\varphi$ preserves norms.

But now using linearity of $\varphi$ and distributivity of the inner product allows us to expand each side of the equation above and gives

$$\|\varphi(x)\|^2 + \|\varphi(y)\|^2 + 2\varphi(x)\varphi(y) = \|x\|^2 + \|y\|^2 + 2xy.$$

Using again the fact that $\varphi$ is norm-preserving and canceling like terms yields our desired result,

$$\varphi(x)\varphi(y) = xy.$$

6.1.1 Matrix of an Orthogonal Transformation

An orthonormal basis allows us to compute the inner product on $V$ by using the standard dot product. We will see now that an orthonormal basis can also help us understand orthogonal transformations on $V$.

Theorem 6.22. Given an orthonormal basis $e = (e_1, \ldots, e_n)$ for $V$ and a linear transformation $\varphi : V \to V$, $\varphi$ is an orthogonal transformation iff $[\varphi]^T[\varphi]_e = I$.

Proof: Note: throughout we will drop $e$ from our notation since the basis $e$ will remain fixed.

Recall than if $A$ and $B$ are matrices and $Ax = Bx$ for all $x \in V$, then $A = B$. Now if $\varphi$ is orthogonal, then for all $x, y \in V$ we have

$$\varphi(x)\varphi(y) = xy.$$

So, in matrix form,

$$[\varphi(x)]^T[\varphi(y)] = [x]^T[y].$$

Recalling that $[\varphi(x)] = [\varphi][x]$ and $(AB)^T = B^T A^T$ allows us to simplify the above to

$$[x]^T[\varphi]^T[\varphi][y] = [x]^T I[y]$$

and now since the equation above holds for all $x$ and $y$, we use the first fact above (twice) to conclude that

$$[\varphi]^T[\varphi] = I.$$
**Exercise 6.23.** The proof given is one implication stated in the theorem. Prove the other.

**Exercise 6.24.** Prove that for matrices $A$ and $B$ $(AB)^T = B^T A^T$.

We now use the result of this theorem as a definition for matrices.

**Definition 6.25.** A matrix $A \in M_n(\mathbb{R})$ is an *orthogonal matrix* if

$$A^T A = I$$

Note that this is the same as saying that the columns of $A$ form an orthonormal system.

**Exercise 6.26.** MAGIC #3: Prove that if the columns of $A$ form an orthonormal system, so do the rows.

Hint: By definition $A^T$ is a left inverse for $A$. By MAGIC #2 a right inverse for $A$ exists. Show that the two must be equal.

**Corollary 6.27.** The matrix $A$ is orthogonal iff $A$ is invertible and $A^T = A^{-1}$.

**Claim 6.28.** If $A$ is orthogonal, $\det(A) = \pm 1$.

**Proof:**

$$1 = \det(I) = \det(A^T A) = \det(A) \det(A^T) = (\det(A))^2.$$  

**Exercise 6.29.** Show that matrices of the form

$$\begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix}$$

are the only $2 \times 2$ orthogonal matrices.

Hint: Recall that the eigenvalues of the first matrix are $\cos \alpha \pm i \sin \alpha$; numbers of this form have unit norm. What are the eigenvalues of the second matrix? Answer this question without calculation.

**Exercise 6.30.** If $\lambda \in \mathbb{C}$ is an eigenvalue of an orthogonal matrix, then $|\lambda| = 1$.

**Corollary 6.31.** If $A$ is an orthogonal $n \times n$ matrix, then $|\det(A)| \leq n$.

Hint: $\det(A) = \sum_{i=1}^n \lambda_i$, where the $\lambda_i$ are the eigenvalues of $A$. 

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Now let $A$ be any matrix, and let $a_1, \ldots, a_n$ be the rows of $A$, so

$$A = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

**Exercise 6.32. (Hadamard’s inequality)** Show that for any matrix $A$

$$|\det(A)| \leq \Pi ||a_i||$$

and furthermore equality holds iff either one of the $a_i$ is zero or the $a_i$ are pairwise orthogonal.

Hint: Recall that $\det(A)$ is the volume of the parallelepiped spanned by the rows of $A$.

**Exercise 6.33.** Prove the Pythagorean theorem in $n$-dimensional real Euclidean space: for orthogonal vectors $a_1, \ldots, a_k$, $|| \sum a_i ||^2 = \sum ||a_i||^2$. 

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