Worksheet on determinants.

Travis Schedler, problems from L. Babai and Proskuryakov's book

Notation: For any function f of i and j, we use the notation

$$(f(i,j)) := \begin{pmatrix} f(1,1) & f(1,2) & \dots & f(1,n) \\ f(2,1) & f(2,2) & \dots & f(2,n) \\ \vdots & & \ddots & \\ f(n,1) & f(n,2) & \dots & f(n,n) \end{pmatrix}.$$

Definition of determinant. $\det(a_{ij}) = \sum_{\sigma \text{ a permutation}} \mathrm{sgn}(\sigma) \prod_i a_{i,\sigma(i)}$, where $\mathrm{sgn}(\sigma) = 1$ is σ is an even permutation and $\mathrm{sgn}(\sigma) = -1$ if σ is an odd permutation.

Prove the following determinant equalities:

1. $\det(\gcd(i,j)) = \prod_{k \leq n} \varphi(k)$, where φ denotes the Euler φ-function (see the Basic Number Theory handout).
2. **The Vandermonde determinant.** $\det(x_i^{j-1}) = \prod_{i<j}(x_j - x_i)$ (here $x_i^0 = 1$ for all i).
3. Similar to that, but "skipping" the $n-1$-st power:
$$\det \begin{pmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-2} & x_1^n \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-2} & x_2^n \\ \vdots & & & \ddots & & \\ 1 & x_n & x_n^2 & \dots & x_n^{n-2} & x_n^n. \end{pmatrix} = (x_1 + x_2 + \ldots + x_n) \prod_{i<j}(x_j - x_i).$$
4. Similar to that, but skipping the 1-st power instead:
$$\det \begin{pmatrix} 1 & x_1^2 & x_1^3 & \dots & x_1^n \\ 1 & x_2^2 & x_2^3 & \dots & x_2^n \\ \vdots & & & \ddots & \\ 1 & x_n^2 & x_n^3 & \dots & x_n^n. \end{pmatrix}$$
$$= (x_1 + x_2 + \ldots + x_n) \prod_{i<j}(x_j - x_i) = x_1 x_2 \cdots x_n (x_1 + x_2 + \ldots + x_n) \prod_{i<j}(x_j - x_i).$$
5. Prove that the following matrix is nonsingular (equivalently, it has nonzero determinant) iff all the a_i are distinct and all the b_j are distinct: $(\frac{1}{a_i + b_j})$.
6. **(The Circulant)** Show that
$$\det \begin{pmatrix} a_0 & a_1 & \dots & a_{n-1} \\ a_1 & a_2 & \dots & a_0 \\ \vdots & & \ddots & \\ a_{n-1} & a_0 & \dots & a_{n-2} \end{pmatrix} = \prod_{\omega \text{ an } n\text{-th root of unity}} \sum_{j=0}^{n} a_j \omega^j.$$

This quantity is defined to be the **Circulant** and is denoted by $C(a_0, a_1, \ldots, a_{n-1})$.

Note: This has a generalization to any abelian group (rather than the cyclic group of order n), replacing roots of unity by characters of the group.

7. For any square matrices A, B of the same size, $\det(AB) = (\det A)(\det B)$.

8. Prove that

$$\det \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ 2 & 2 & 3 & \ldots & n \\ 3 & 3 & 3 & \ldots & n \\ \vdots & & & \ddots & \\ n & n & n & \ldots & n \end{pmatrix} = (-1)^{n-1} n.$$

9. Prove that

$$\det \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \binom{2}{1} & \binom{3}{1} & \ldots & \binom{n}{1} \\ 1 & \binom{3}{2} & \binom{4}{2} & \ldots & \binom{n+1}{2} \\ \vdots & & & \ddots & \\ 1 & \binom{n}{n-1} & \binom{n+1}{n-1} & \ldots & \binom{2n-2}{n-1} \end{pmatrix} = 1.$$

10. Prove $\det(a_i b_j) = a_1 b_n \prod_{i=1}^{n-1} (a_{i+1} b_i - a_i b_{i+1})$.

11. Prove

$$\det \begin{pmatrix} 1 & \cos\alpha_1 & \cos 2\alpha_1 & \ldots & \cos(n-1)\alpha_1 \\ 1 & \cos\alpha_2 & \cos 2\alpha_2 & \ldots & \cos(n-1)\alpha_2 \\ \vdots & & & \ddots & \\ 1 & \cos\alpha_n & \cos 2\alpha_n & \ldots & \cos(n-1)\alpha_n \end{pmatrix} = 2^{(n-1)^2} \prod_{1 \le i < k \le n} \sin \frac{\alpha_i + \alpha_k}{2} \sin \frac{\alpha_i - \alpha_k}{2}.$$

12. Prove that

$$\det \begin{pmatrix} \cos\alpha & 1 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 2\cos\alpha & 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 2\cos\alpha & 1 & \ldots & 0 & 0 \\ \vdots & & & & \ddots & & \\ 0 & 0 & 0 & 0 & \ldots & 1 & 2\cos\alpha \end{pmatrix} = \cos n\alpha.$$

13. Do problems 279–284 in the Proskuryakov handout.

14. Do problems 383, 388, 390, 391, and 392 in the handout.

Definition. A $\mathbf{k} \times \mathbf{l}$**-submatrix** is obtained by choosing k rows and l columns, and taking the kl entries in the intersection to form a $k \times l$-matrix.

Definition. A $k \times k$-minor is the determinant of a $k \times k$-submatrix.

Rank. Recall that for any square matrix A, the **rank**, rk A, is defined to be the largest nonnegative integer k such that there is a nonzero $k \times k$-minor of the matrix (where $k = 0$ if it is the zero matrix).

1. Prove that the rank is also the greatest number of columns that are linearly independent (similarly, the greatest number of rows). Linearly independent means that no nontrivial linear combination of them gives the zero vector.

2. Find the rank as a function of λ:

$$\begin{pmatrix} 3 & 1 & 1 & 4 \\ \lambda & 4 & 10 & 1 \\ 1 & 7 & 17 & 3 \\ 2 & 2 & 4 & 3 \end{pmatrix}.$$

3. Find the rank as a function of λ:

$$\begin{pmatrix} 1 & \lambda & -1 & 2 \\ 2 & -1 & \lambda & 5 \\ 1 & 10 & -6 & 1 \end{pmatrix}.$$

Smith Normal Form. For any rectangular matrix A with integer coefficients, we define the "Smith Normal Form" to be the diagonal matrix

$$\begin{pmatrix} d_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & \ddots & & & & \\ 0 & 0 & \dots & d_n & 0 & \dots & 0 \end{pmatrix}, \tag{1}$$

such that $d_1 d_2 \cdots d_i$ is the (nonnegative) gcd of all $i \times i$-minors of the matrix.

1. Prove that all d_i are integers.

2. Show that any matrix can be reduced to a diagonal matrix by the following row and column operations:

   (a) Adding a multiple of one row to another;

   (b) Adding a multiple of one column to another.

3. Show that, in addition, the diagonal matrix thus obtained can be of the form (1) where

$$d_i \mid d_{i+1}, \forall 1 \leq i \leq n-1. \tag{2}$$

4. Prove that row and column operations as defined above do not change the determinant (you have probably used this fact already).

5. Deduce that the matrix of the form (1), (2) obtained by row and column operations is in Smith Normal Form once absolute values are taken of all entries.

6. Deduce that the Smith Normal Form satisfies $d_i \mid d_{i+1}$ for all i (note that this is not immediately obvious from the definition.)

7. Show that the matrix of the form (1), (2) obtained by row and column operations is unique up to changing an even number of signs. In other words, the matrix contains two pieces of information: the Smith Normal Form and the sign of the determinant.

Characteristic Polynomial. For any square matrix A, we define the characteristic polynomial $K_A(\lambda)$ to be $\det(A - \lambda I)$ where I is the identity matrix, and λ is the variable.

1. Prove that $K_A(\lambda)$ is a polynomial of degree n of λ.

2. Compute the characteristic polynomial of the matrix of 1's, i.e.

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & & \ddots & \\ 1 & 1 & \dots & 1 \end{pmatrix}.$$

3. Show that if $\lambda = \lambda_0$ is a root of the characteristic polynomial iff $A - \lambda_0 I$ is singular.

4. Define an **eigenvalue** λ_0 of A to be a number such that $Av = \lambda_0 v$ for some column vector v. Show that any eigenvalue is a root of the characteristic polynomial.

5. Deduce that any $n \times n$-matrix has at most n eigenvalues.

6. Define a **symmetric minor** to be a minor whose choice of rows are the same as the choice of columns (i.e. the diagonal of the minor is a subset of the diagonal of the matrix). Prove that the polynomial is of the form $K_A(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_n$, where a_i is the sum of all symmetric $i \times i$-minors, multiplied by $(-1)^i$.

   Definition. The trace of a square matrix is defined by $\operatorname{tr}(a_{ij}) = \sum_i a_{ii}$, i.e. the sum of the diagonal entries.

7. Deduce that a_n is $(-1)^n$ times the determinant of A, and a_1 is the negative of the trace (the sum of the diagonal entries.)

8. Prove that if A has integer coefficients, then a_i is a multiple of $d_1 d_2 \cdots d_i$ where (d_i) is the Smith Normal Form of the matrix.

9. Back to the case where $(\lambda - \lambda_0) \mid K_A(\lambda)$ (i.e. $A - \lambda_0 I$ is singular). Show that the multiplicity k of the root λ_0 (the greatest $k \geq 1$ such that $(\lambda - \lambda_0)^k \mid K_A(\lambda)$) is equal to $n - \operatorname{rk} \operatorname{SN}(A - \lambda_0 I)$, where $\operatorname{SN}(B)$ gives the Smith Normal Form of B. Hint: Start with the case $\lambda_0 = 0$, and recall the definition of rank from this worksheet (for a diagonal matrix, the rank becomes a very simple thing!).