REU 2005 · Discrete Mathematics · Lecture 5

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1 Lecture 5

1.1 Proof of Cayley's Formula cont'd

If $A = (a_{ij})$ is a triangular, i.e., $(\forall i < j)(a_{ij} = 0)$ then $\det(A) = \Pi a_{ii}$ (product of diagonals).

To complete the proof of Cayley's Formula, we need the following theorem.

Theorem 1.1.

$$\left| \begin{pmatrix} x & y & y & \dots & y \\ y & x & y & \dots & y \\ \vdots & & \ddots & & \\ y & y & y & \dots & x \end{pmatrix} \right| = (x-y)^{n-1}(x+(n-1)y) \tag{1.1}$$

One way to compute the determinant is to subtract each row from the previous row. That is

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**for** $i = 1$ to $(n-1)$
**do** $r_i = r_i - r_{i+1}$.
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So we get

$$\begin{pmatrix} x-y & y-x & 0 & \dots & 0 \\ 0 & x-y & y-x & \dots & 0 \\ \vdots & & \ddots & & \\ y & y & y & \dots & x \end{pmatrix} \tag{1.2}$$

Note that if we first factor out $(x-y)$ from each row except the last one, and then add each column to the next column, we get

$$(x-y)^{n-1} \det \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & & \\ y & 2y & 3y & \dots & x+(n-1)y \end{pmatrix}, \tag{1.3}$$

This ia a triangular matrix. So the determinant is $(x+(n-1)y)(x-y)^{n-1}$.

Another idea: Add all rows to the last row, factor out $(x+(n-1)y)$ from the last row and then delete yr_n (y time the nth row) from all the other rows. Then we get

$$(x+(n-1)y)\det\begin{pmatrix} x-y & 0 & 0 & \dots & 0 \\ 0 & x-y & 0 & \dots & 0 \\ \vdots & & \ddots & & \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix} \tag{1.4}$$

again a triangular matrix and the determinant equals $(x+(n-1)y)(x-y)^{n-1}$.

Cayley's Formula

Recall that the number of trees in a complete undirected graph on n vertices is the determinant of the reduced laplacian.

Now the reduced laplacian is a $(n-1)\times(n-1)$ matrix and looks like the following matrix:

$$\begin{pmatrix} x & y & y & \dots & y \\ y & x & y & \dots & y \\ \vdots & & \ddots & & \\ y & y & y & \dots & x \end{pmatrix} \tag{1.5}$$

where $x=(n-1)$ and $y=-1$.

From the theorem we know that the determinant of the matrix is $(x+(n-1)y)(x-y)^{n-1}$. Plugging the value of x and y we get the number of trees on a given set of n vertices is $((n-1)-(n-2))(n-1-(-1))^{n-2}=n^{n-2}$ (Cayley's Formula). Note that no really simple proof of this formula exists (which one might hope for, given the simple answer).

1.2 Ideals in Semigroups

Recall the definition:

Definition 1.2. Let S be a semigroup. I is an *ideal* of S (denoted $I \triangleleft S$) if $SI \subseteq I$ and $IS \subseteq I$.

Last time, we had the following exercise:

Exercise 1.3. *$IJ \neq \emptyset$ if I and J are nonempty ideals.*

Proof. Now if $I, J \triangleleft S$ and $\exists a \in I, \exists b \in J$, then we want to claim $ab \in I \cap J$. This follows because I is in particular a right ideal, and J is in particular a left ideal. ☐

In fact, the preceding argument shows that

Corollary 1.4. *$IJ \subseteq I \cap J$.*

Recall that the exercise immediately implied that any finite nonempty semigroup has an unique minimal ideal, dubbed the "core." Now let's prove:

Exercise 1.5. *If $S \neq \emptyset$ is a finite, commutative semigroup, then core(S) is a group. (In particular, since we have already proved that core(S) is an ideal, what remains to be shown is that every element of the core has an inverse in the core as well).*

Proof. Let $I = \text{core}(S)$.

Recall that we previously proved that a nonempty semigroup in which every linear equation ($ax = b$, $ya = b$) is solvable is a group. Since we are dealing with a commutative semigroup, we only have to deal with linear equations with multiplication on the right. So, we will be done if we can prove the following claim:

Claim: $(\forall a, b \in I)(\exists x \in I)(ax = b)$.

Recall the notation $aI = \{ax : x \in I\}$. The claim is equivalent to saying that $(\forall a \in I)(aI = I)$. It's clear that $aI \subseteq I$ because $I \triangleleft S$. But also $aI \triangleleft S$ because I is in particular a right ideal, so $aIb \subset aI$ for all $b \in S$, implying that aI is a right ideal. By commutativity, aI is a two-sided ideal. Now, since I is minimal, $aI \subseteq I \implies I = aI$.

Therefore, every $b \in I$ belongs in aI, i.e., b can be written as $ax, x \in I$, i.e., the equation $ax = b$ is solvable.

□

1.3 The Chip-firing Game

We have a digraph $\mathcal{G}$, and on each vertex of $\mathcal{G}$ we have a stack of chips. In the context of chip-firing games, we will refer to the digraph $\mathcal{G}$ as the *ambient space*.

A *configuration* is a function $f : V \to \mathbb{N}$ (where V is the set of vertices), which tells for each vertex how many chips are on that vertex.

Definition 1.6. An *unstable* vertex v is one such that $f(v) \geq \deg^+(v)$. A configuration is called *stable* if no vertex in unstable.

Unstable vertices can "fire": this means they send a chip along every edge starting at v, thus lowering the number of chips on v by $\deg^+(v)$, and raising the number of chips on every other vertex w by the number of arcs from v to w.

A sequence of such firings is called an "avalanche." The following surprising theorem about avalanches is fundamental to the theory of chip-firing games:

Theorem 1.7.

1. *If an avalanche starting at a configuration C never terminates then no avalanche starting at C terminates.*
2. *If two avalanches take the same configuration C to stable configurations C_1 and C_2, then $C_1 = C_2$. Thus, the final, stable configuration depends only on the initial configuration.*

Definition 1.8. The *score vector* of an avalanche is a vector $s : V \to \mathbb{N}$ which says the number of times each vertex fired in the course of the avalanche.

Theorem 1.9. *If two avalanches take the same configuration C to stable configurations, then they must have the same score vector.*

Thus the score vector of an avalanche which ends in a stable configuration depends on the initial configuration and not the order of firings. Theorem 1.9 implies Theorem 1.7, but not *vice versa*.

Exercise* 1.10. *Prove the following theorem (due to G.Tardos):*

If the ambient space is ***undirected*** *then the length of any avalanche is either infinite or $< c \cdot (n) \cdot (diam)(\#edges)$, for some fixed constant c ($c = 4?$), where diam is the diameter of the graph.*

Specifically, if $\mathcal{G}$ does not have parallel edges, then we get that the length of the avalanche is either infinite or is less than cn^4.

Exercise 1.11. *Construct a directed graph (without parallel edges) on which the avalanche can be exponentially long. (I.e., there exists an avalanche with finite length, but whose length is greater than c^n for some constant c, where n is the number of vertices in $\mathcal{G}$. In order for this to really make sense, you should construct an infinite family of graphs, one for each n, in which each graph has an avalanche of finite length greater that c^n, for some constant $c > 1$ which does not depend on n.)*

1.4 The Abelian Sandpile Model

The Abelian Sandpile Model is like the chip-firing game, except that there is a special vertex designated as the **sink** which swallows chips and never fires.

We assume that the sink is reachable from any vertex.

Exercise 1.12. *Every avalanche is finite!*

Let $\mathcal{M}$ be the set of stable configurations. By the previous exercise, we have a map $\sigma : \mathbb{N}^{V\setminus\{\text{sink}\}} \to \mathcal{M}$ taking any arbitrary configuration to a stable one. Applying σ to a configuration will be referred to as *stabilizing* the configuration, and $\sigma(h)$ will be called the *stabilization* of h.

We can define an "addition" operation on $\mathcal{M}$: we add the two stable configurations pointwise and then apply σ. That is, for $f, g \in \mathcal{M}$, we have $f \oplus g := \sigma(f + g)$ where the usual $+$ just adds chips at each vertex (i.e. the usual addition on $\mathbb{N}^{V\setminus\{\text{sink}\}}$).

Exercise 1.13. *Show that $\mathcal{M}$ is a monoid under $\oplus$. Hint: $f(v) = 0$ is the identity; commutativity is obvious; associativity follows from the fact that $(f \oplus g) \oplus h = \sigma(f + g + h)$ (prove this fact).*

Definition 1.14. Let $\mathcal{G}$ be the unique minimal ideal of $\mathcal{M}$. Then $\mathcal{G}$ is called the *sandpile group*. This group is abelian because $\mathcal{M}$ is commutative.

Theorem 1.15. $|\mathcal{G}| = \det(\Delta) =$ *the number of spanning trees oriented to the sink, where δ is the reduced Laplacian (the row and column correspomding to the sink are deleted).*

Since the number of spanning trees is the determinant of the reduced Laplacian of the digraph, we see that the reduced Laplacian is related to the study of $\mathcal{G}$. In fact, we will find that the structure of $\mathcal{G}$ is closely related to the reduced Laplacian, not just to its determinant.

Definition 1.16. Elements of $\mathcal{G}$ are called *recurrent configurations.* All other stable configurations are called *transient.*

Note that the 0 configuration, which is the identity element of $\mathcal{M}$, is not recurrent for most sandpiles.

Exercise 1.17. *1.* $0 \in \mathcal{G} \iff \mathcal{M} = \mathcal{G}$ *(all stable states are recurrent.*

2. Construct those ambient spaces which result in $\mathcal{M} = \mathcal{G}$.

Therefore, the identity in $\mathcal{G}$ will generally be something more complex. We saw the diagram for the identity of the 200×200 grid sandpile.

1.5 Group Quotients

Definition 1.18. Let G be a group, and let H be a subgroup of G. For each $a \in G$, we define a *right coset of* H (denoted Ha) to be the set $Ha := \{ha \mid h \in H\}$.

Definition 1.19. (Quotient Group) Let G be an abelian group, $H \leq G$ a subgroup consider the set of cosets $G/H = \{Ha \mid a \in G\}$. Define $(Ha) \cdot (Hb) = Hab$.

Exercise 1.20. *Hab depends only on the cosets Ha, Hb, and not on the choices of representative a and b.*

So G/H is an abelian group.

Now let T be the set of rows of the reduced Laplacian $\subset \mathbb{Z}^{n-1}$. Then $\langle T \rangle = \Lambda$ is a lattice.

Exercise* 1.21. *Prove that* $\mathcal{G} \cong \mathbb{Z}^{n-1}/\Lambda$.

For the solution of the above exercise we will need to know the following:

Exercise 1.22. *If $\mathcal{M}$ is a finite commutative monoid and $\mathcal{G}$ is its minimal ideal, then*

1. *There exists $h : \mathcal{M} \twoheadrightarrow \mathcal{G}$ is an onto homomorphism, $h(ab) = h(a)h(b)$.*

$$\begin{tikzcd} \mathcal{M} \arrow[r, "a \mapsto h(a)"] & \mathcal{G} \end{tikzcd} .$$

2. *$\mathcal{G}$ is the* ***largest*** *group onto which $\mathcal{M}$ has a homomorphism. That is, if $k : \mathcal{M} \to \mathcal{H}$ is a homomorphism, then there exists a unique $\ell : \mathcal{G} \to \mathcal{H}$ such that $k = \ell h$. The situation is summarized in this diagram:*

$$\begin{tikzcd} \mathcal{M} \arrow[r, "h"] \arrow[dr, "k"'] & \mathcal{G} \arrow[d, dashed, "\ell"] \\ & \mathcal{H} \end{tikzcd} ,$$

1.6 Generators and Relations in Commutative Monoids

Since we will be working in commutative monoids, we will use additive notation.

We say that the monoid $\mathcal{M} = \langle a, b, c\rangle_{\text{monoid}}$ is generated by $\{a, b, c\}$, which means that the elements of M are linear combinations of these three elements with nonnegetative integer coefficients.

Notation 1.23. Let $\mathcal{M}$ be a commutative monoid, written in additive notation. For a nonnegative integer n, and $a \in \mathcal{M}$, we use $n \cdot a$ to denote the result obtained by adding a to itself n times.

Relations are identities of the form $x_1a + y_1b + z_1c = x_2a + y_2b + z_2c$, where x_i,y_i,z_i are nonnegative integers.

Let E be a set of equations over $\mathbb{N}$ involving symbols $a_1, \ldots, a_k$. There exists a unique "largest" monoid $\mathcal{M} = \langle a_1, \ldots, a_k\rangle$ such that in $\mathcal{M}$, E holds among the a_i. That is,

1. $\mathcal{M} = \langle a_1, \ldots, a_k\rangle$, E holds, and
2. if $\mathcal{N} = \langle a'_1, \ldots, a'_{t'}\rangle$, and E holds, then $\exists f : \mathcal{M} \twoheadrightarrow \mathcal{N}$ such that $f(a_i) = a'_i$.

This allows us to define a monoid using generators and relations:

Definition 1.24. The *monoid generated by the a_i subject to the relations E*, denoted $\mathcal{M} = \langle a_1, \ldots, a_k \mid E\rangle$, is the unique object $\mathcal{M}$ given above.

1.7 The Sandpile Monoid

Notation 1.25. Let $V_0 = V \setminus \{sink\}$.

The "standard" generators of the sandpile monoid are $t_v := (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the v-component, for each $v \in V_0$.

Exercise 1.26.

$$\deg^+(i) \cdot t_i = \sum_{i \in V_0} a_{ij} \cdot t_j. \tag{1.6}$$

Exercise* 1.27. *The sandpile monoid is generated by the t_i subject to the $n - 1$ relations described in Equation(1.6).*

As a consequence, the sandpile group is defined by

$$\mathcal{G} = \langle t_i : i \in V_0 \mid \text{Equations(1.6), for } i \in V_0\rangle,$$

where we interpret the generators and relations in the context that $\mathcal{G}$ is an abelian group.

Since $\mathcal{G}$ is abelian, these equations can be written as

$$(\deg^+(i) - a_{ii}) \cdot t_i - \sum_{j \in V_0, j \neq i} a_{ij} \cdot t_j = 0.$$

Exercise 1.28. $|\mathbb{Z}^{n-1} : \Lambda| = \det(\Delta)$, *where* Λ *is the lattice defined by the reduced Laplacian* Δ *of a sandpile.*

Exercise 1.29. *Infer from this that* $\mathcal{G} \cong \mathbb{Z}^{n-1}/\Lambda$.

Exercise 1.30. *Let* $v_1, \ldots, v_n \in \mathbb{Z}^n$ *be linearly independent. If* $H := \langle v_1, \ldots, v_n \rangle = \sum_{i=1}^{n} \mathbb{Z} v_i$, $H \leq \mathbb{Z}^n$, *then* $|\mathbb{Z}^n : H| = |\det(v_1, \ldots, v_n)|$.

Exercise 1.28 is a special case of the previous exercise.